LECTURE 33
MATH 256A

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We will continue establishing the relationship between schemes and classical varieties.

1. Sober spaces

Recall a space is sober iff for every irreducible closed subset $Z \subseteq X$ there exists some $x \in X$ such that $Z = \overline{\{x\}}$. For any space X we will define:

$$\operatorname{Sob}(X) = \{Z \subseteq X \mid Z \text{ is irreducible and closed}\} \ .$$

Then we give this the topology where the closed sets are

$$V(Y) = \{Z \in \operatorname{Sob}(X) \mid Z \subseteq Y\}$$

for $Y \subseteq X$ closed, which we saw last time is an actual topology. There is a map $i : X \to \operatorname{Sob} X$ which sends $x \mapsto \overline{\{x\}}$. Now the main point is that under this map,

$$i^{-1}(V(Y)) = Y \ .$$

This has an easy consequence, which says that we have

$$\operatorname{Cl}(X) \underset{i^{-1}}{\overset{V}{\rightleftarrows}} \operatorname{Cl}(\operatorname{Sob}(X))$$

where V is surjective, and hence a bijection, and i^{-1} a bijection as well. So somehow the topology is unchanged. In particular, these spaces have exactly the same sheaf theory. In this setting we say that i is a quasi-homeomorphism. I.e. it is a homeomorphism which doesn't care about being bijective on the underlying points. This means it is a bijection of closed, open, and constructible subsets. It also means that the maps

$$i_* : \mathbf{PreSh}(X) \to \mathbf{PreSh}(\operatorname{Sob}(X)) \qquad i_* : \mathbf{Sh}(X) \to \mathbf{Sh}(\operatorname{Sob}(X))$$

are isomorphisms of categories.

This is also (as we would hope) a sober space. We know irreducible closed subsets $Y \subseteq \operatorname{Sob}(X)$ are in correspondence with irreducible closed subsets $Z \subseteq X$, i.e. $Y = V(Z)$. But $V(Y) \supseteq V(Z)$ iff $Y \supseteq Z$ iff $Z \in V(Y)$, which is saying $V(Z) = \overline{\{Z\}}$ in $\operatorname{Sob}(X)$, so $\operatorname{Sob}(X)$ is sober.

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For any map $f : X \to X'$, we can map $x \mapsto \overline{f(x)}$ which maps $\operatorname{Sob}(X) \to \operatorname{Sob}(X')$ which makes Sob into a functor:

$$\begin{array}{ccc} \operatorname{Sob}(X) & \xrightarrow{\operatorname{Sob}(f)} & \operatorname{Sob}(X') \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle i'} \\ X & \xrightarrow{\quad f \quad} & X' \end{array}$$

Any sober W factors as:

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & W \\ \downarrow & \nearrow & \\ \operatorname{Sob}(X) & & \end{array}$$

This is saying that $\operatorname{Sob} : \mathbf{Top} \to \operatorname{Sob}$ is left adjoint to $\operatorname{Sob} \hookrightarrow \mathbf{Top}$. Even better is that if f is a quasi-homeomorphism, so is $\operatorname{Sob}(f)$ of course, but it will even be an actual homeomorphism.

Any functor $\mathbf{Top} \to \mathbf{C}$ which sends quasi-homeomorphisms to isomorphisms will factor uniquely through Sob. This is really saying that Sob is equivalent to the category $Q^{-1}\mathbf{Top}$ where we have formally inverted quasi-homeomorphisms.

Remark 1. This was all supposed to show us that the topology on a scheme might seem weird, but actually it's very natural. Every topological space ought to be like this.

2. Jacobson sober spaces

Now we want to ask the question of which sober spaces we get if we start with a T_1 space. First we should notice that in any T_1 space[1] the points such that $\{x\} = \overline{\{x\}}$ as just being the minimal closed subsets (which are automatically also irreducible).

In any case, when X is T_1, if we look at the map $i : X \to \operatorname{Sob}(X)$, $i(X) = \operatorname{Sob}(X)_{\mathrm{cl}}$. So we can somehow recover the space X from $\operatorname{Sob}(X)$. But to relate T_1 spaces to sober spaces, we have to see which sober spaces come from T_1 spaces.

Definition 1. A sober space is Jacobson if every closed $Z \subseteq X$ has $Z = \overline{Z_{\mathrm{cl}}}$

So there are sort of enough closed points so you can see all the closed subsets using just the closed points. We will see later that this notion is related to the notion of a Jacobson radical of an ideal.

Example 1. Any T_1 space is trivially Jacobson.

Example 2 (Non-example)**.** For any local ring R, $\operatorname{Spec} R$ is not Jacobson since it has only one closed point, but many generic points of other irreducible closed subsets. In particular, many closed subsets which aren't this one point.

Lemma 1. *TFAE:*

- a. *X is Jacobson.*
- b. *$Z = \overline{Z_{cl}}$ for Z irreducible closed subspaces of X.*
- c. *$X_{cl} \hookrightarrow X$ induces $\operatorname{Cl}(X) \hookrightarrow \operatorname{Cl}(X_{cl})$, or $\operatorname{Op}(X) \hookrightarrow \operatorname{Op}(X_{cl})$, or $X_{cl} \hookrightarrow X$ is a quasi-homeomorphism.*

[1] And maybe T_0, but Professor Haiman is nervous about some pathological examples preventing this from being true.

d. $\operatorname{Sob}(X_{cl}) \xrightarrow{\sim} \operatorname{Sob}(X)$ *is bijective (so a homeomorphism).*
e. $\operatorname{Sob}(X)$ *is Jacobson.*
f. Every locally closed $Z \subseteq X$ *has* $Z \cap X_{cl} \neq \emptyset$ *(or for* Z *irreducible).*

We have the maps:

$$\mathbf{T_1} \underset{(-)_{\mathrm{cl}}}{\overset{\mathrm{Sob}}{\rightleftarrows}} \mathbf{Jac}$$

and then we can ask what the image of Sob is, which tells us that the right definition of the category **Jac** is Jacobson spaces and closed point preserving maps.

Fact 1. *Open, closed, locally closed subsets* $Y \subseteq X$ *are Jacobson iff* X *is.*

3. Jacobson rings

Theorem 1. *i. Every scheme* X *locally of finite type over a field* k *(algebraic scheme) is Jacobson.*
ii. Every k*-morphism of algebraic schemes over* k *sends* X_{cl} *to* Y_{cl}

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ & \searrow \quad \swarrow & \\ & \operatorname{Spec} k & \end{array}$$

iii. If X *is algebraically over* k *(not necessarily algebraically closed), then* $k(x)$ *is a finite algebraic extension of* k*.*

Remark 2. For k algebraically closed, this theorem tells us that the residue fields are all k.

Remark 3. The third part is somehow a version of the nullstellensatz.

We will just define the following:

Definition 2. A ring R is Jacobson iff $\operatorname{Spec} R$ is Jacobson.

This somehow means that for any subset $F \subseteq \operatorname{Spec} R$, tautologically:

$$\overline{F} = V\left(\bigcap_{P \in F} P\right)$$

We know the closed subsets are $Z = V(I)$ for arbitrary radical ideals I, and then

$$Z_{\mathrm{cl}} = \{\mathfrak{m} \mid I \subseteq \mathfrak{m}\}$$

and then

$$\overline{Z_{\mathrm{cl}}} = V\left(\bigcap_{\mathfrak{m} \supseteq I} \mathfrak{m}\right)$$

so for $Z = \overline{Z_{\mathrm{cl}}}$, we have that

$$I = \bigcap_{\mathfrak{m} \supseteq I} \mathfrak{m}$$

This is all reversible. I.e. a ring R is Jacobson iff for every radical ideal I, I is equal to its Jacobson radical:

$$I = \bigcap_{\mathfrak{m} \supseteq I} \mathfrak{m}$$