LECTURE 34

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We will continue with our general program of understanding how classical algebraic varieties over $k = \overline{k}$ are in some sense the same as reduced schemes locally of finite type over k. Basically the important bit is seeing this for affine varieties, and then things glue together fine. So taking an affine variety, there is some ring of functions, and then the scheme we want to associate to this is Spec of that ring. For $X \subseteq k^n$ and $Y = \operatorname{Spec}(\mathcal{O}(X))$ then $x \in X$ gives us a map $\mathrm{ev}_x : \mathcal{O}(X) \to k$ which maps $f \mapsto f(x)$ which gives us a map $\operatorname{Spec} k \to Y$. So $X \simeq Y(k)$ the set of k points of Y thought of as a scheme over k. These are closed points, but what's not so obvious is that these are all of the closed points. But if we knew that, and if we also know that every subset of $\operatorname{Spec} \mathcal{O}(X)$ is the closure of the closed points in it (i.e. that it is Jacobson) then the classical variety X (which is T_1) has an associated space $\mathrm{Sob}(X)$. Then we can say $\operatorname{Spec}(\mathcal{O}(X))$ is just $\mathrm{Sob}(X)$.

So we want to prove the following:

Theorem 1. *If X is locally of finite type over a field k, then X is Jacobson, and for every closed point $x \in X$ the residue field $k(x)$ is a finite algebraic extension of k.*

We also want to say that k-morphisms always send closed points to closed points.

1. Locally of finite type

What we want is that if we have a morphism of rings: $A \to B$ such that B is finitely generated as an A-algebra:

$$B = A[x_1, \cdots, x_n]/J$$

then we want to say that $\operatorname{Spec} B$ is somehow of "finite type" over $\operatorname{Spec} A$, and $f : \operatorname{Spec} B \to \operatorname{Spec} A$ is a morphism of finite type. But it's not clear that B being a f.g. A-algebra is a local condition.

Definition 1. A morphism $f : X \to Y$ is *locally of finite type* if for all open affine $\operatorname{Spec} B = U \subseteq X$ and $\operatorname{Spec} A = V \subseteq Y$ such that $f(U) \subseteq V$, so $f|_U : U \to V$, and then this corresponds to a ring homomorphism $A \to B$, and we require B to be a finitely generated A-algebra.

Remark 1. This definition is good and bad. It's somehow weakly local since if $U \subseteq X$ open then

$$\begin{array}{ccc} U & \hookrightarrow & X \\ \downarrow & & \downarrow \\ V & \hookrightarrow & Y \end{array}$$

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and $U \to V$ is locally of finite type. But it's bad in the sense that you can't a priori check this on a covering, so it's not somehow strongly local. For example, if B is a finitely generated A-algebra, it is not at all obvious that $\operatorname{Spec} B \to \operatorname{Spec} A$ is locally of finite type. But we can justify this definition if we can show that these non-obvious things are actually true.

This definition is equivalent to saying that for every affine $\operatorname{Spec} A \subseteq Y$, the preimage is locally of finite type.

Proposition 1. *Let $f : X \to \operatorname{Spec} A$. There exist open affines $X = \bigcup_\alpha X_\alpha$ for $X_\alpha = \operatorname{Spec} B_\alpha$ such that B_α is finitely generated over A iff f is locally of finite type.*

Proof. The backwards direction is trivial. Let $\operatorname{Spec} B = U \subseteq X$ be open. If $f(U) \subseteq V \subseteq \operatorname{Spec} A$ where $V \subseteq \operatorname{Spec} A$ is open, then this gives us a map of rings $A \to A' \to B$ for some A'. If B is f.g. over A, this is also f.g. over A' with the same generators. So it is enough to check this on a cover. Now cover each $U \cap X_\alpha$ by affines $(X_\alpha)_g = \operatorname{Spec} B_\alpha \left[g^{-1}\right]$ for $g \in B_\alpha$. Now we can cover each $(X_\alpha)_g$ by opens $U_h = \operatorname{Spec} B \left[u^{-1}\right]$. In general, if we have $U \cap$ any other open subset, and we want to know how U_h intersects with this subset, it will be the intersection sub h:

$$\begin{array}{ccc} V_h & \longrightarrow & V \\ \downarrow & & \downarrow \\ U_h & \hookrightarrow & U \end{array}$$

In this case this means

$$U_{gh} = (X_\alpha)_{gh}$$

Since B_α is a f.g. A-algebra, then

$$B\left[h^{-1}\right] \simeq B_\alpha \left[(gh)^{-1}\right]$$

is a f.g. A-algebra.

Then we have

$$U = \bigcup_{i=1}^{n} U_{h_i}$$

and since these generate the unit ideal,

$$1 = a_1 h_1 + \cdots + a_n h_n$$

and the $B\left[h_i^{-1}\right]$ are f.g. over A, and WLOG, this is generated by some elements h_i^{-1} and $x_{ij} \in B$.

So let $B' \subseteq B$ be the subalgebra generated by the h_i, a_i, and x_{ij}. The h_i still generates the unit ideal in B' (since we included the a_i). If we let $U' = \operatorname{Spec} B'$, then we can say that the $U'_{h_i} = \operatorname{Spec} B' \left[h_i^{-1}\right]$ cover U'. But now $B\left[h_i^{-1}\right] = B'\left[h_i^{-1}\right]$, which means their associated sheaves are equal, so the sheaves $\tilde{B}' = \tilde{B}$ on U'_{h_i}. But since the B'_{h_i} cover, $B' = B$. ∎

Corollary 1. *If we have $\operatorname{Spec} B \to \operatorname{Spec} A$ this is locally of finite type iff B is f.g. over A.*

Corollary 2. *If $U \subseteq \operatorname{Spec} A$ is open, (affine or not) then $U \to \operatorname{Spec} A$ is locally of finite type.*

Corollary 3. *If $j : X \hookrightarrow Y$ is an open embedding, then j is locally of finite type.*

Corollary 4. *If $i : X \to Y$ is an open embedding, then j is locally of finite type.*

Proof. If $X \to Y$ is a closed embedding, then for $U = \operatorname{Spec} A \subseteq Y$ then $i^{(U)} = \operatorname{Spec}(A/I)$. □

It is not yet obvious that compositions of maps locally of finite type are locally of finite type. We will continue with this next time...