LECTURE 35
MATH 256A

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1. Locally of finite type

Recall last time we defined a morphism $f : X \to Y$ to be locally of finite type (lft) if whenever we take an affine open subset $U = \operatorname{Spec} A \subseteq Y$ and an affine open set $V = \operatorname{Spec} B \subseteq X$ such that $f(V) \subseteq U$ V is f.g. over A. This is not a priori a local condition, but we do have the following:

Proposition 1. *Let $f : X \to \operatorname{Spec} A$ where X is covered by affine $X_\alpha = \operatorname{Spec} B_\alpha$. If B_α f.g. over A then f is locally of finite type.*

2. Corollaries

The previous proposition gives us a bunch of corollaries:

Corollary 1. *Open $\operatorname{Spec} B \subseteq \operatorname{Spec} A$ implies B is finitely generated over A.*

Corollary 2. *Let $f : X \to Y$ where X is covered by open U_α. Then if $U_\alpha \to Y$ is locally of finite type for all α, then f is locally of finite type.*

Proof. For each open affine in Y we just need to show its preimage is locally of finite type, then cover its preimage with affines, which we can take as being contained in the U_α, and since these are f.g. over Y, the proposition gives us the result. □

Corollary 3. *If $f : X \to Y$ and $g : Y \to Z$ such that f and g are lft then $g \circ f$ is lft.*

Proof. Pick one W, then we get a bunch of Us and Vs:

$$\begin{array}{ccccc} X & \longrightarrow & Y & \longrightarrow & Z \\ \uparrow & & \uparrow & & \uparrow \\ \operatorname{Spec} C = U & & \operatorname{Spec} B = V & & \operatorname{Spec} A = W \end{array}$$

Then B is f.g. over A and C is f.g. over B, so C is f.g. over A and the proposition gives us the result. □

Now we want to say this property is local on both the codomain and the domain.

Corollary 4. *Given a morphism $f : X \to Y$, suppose that $\forall x \in X$, there exists an open neighborhood $x \in V \subseteq X$ and a open neighborhood $f(V) \subseteq U \subseteq Y$ such that $V \to U$ is lft, then f is lft.*

Proof. Compose $V \to U \hookrightarrow Y$, then by the previous corollary, the $X \to Y$ are lft, so we are done. □

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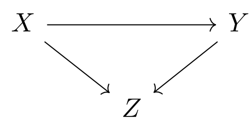
Corollary 5. *Given $f : X \to Y$ and $g : Y \to Z$ such that $f \circ g$ is lft, then f is lft.*

Proof. Choose subsets as in the proof of corollary 3. Now we have C is a f.g. A-algebra, but then it a f.g. over B with the same generators, so f is lft. □

A better way to think of this is that X and Y are schemes over Z, and then we have a morphism of these as schemes over Z



So the previous corollary is saying that for $X \to Z$ lft, any morphism of schemes over Z is lft.

3. General nullstellensatz

Now we finally meet the true story of the nullstellensatz:

Theorem 1. *If $f : X \to Y$ is lft and Y is Jacobson then:*

i. *X is Jacobson.*
ii. *$f(X_{cl}) \subseteq Y_{cl}$, i.e. f sends closed points to closed points.*
iii. *For all closed points $x \in X_{cl}$ the residue field of x is a finite algebraic extension $K(y) \hookrightarrow K(x)$ where $y = f(x)$.*

Remark 1. The last part of the theorem is effectively equivalent to the nullstellensatz. This is a classic example of Grothendieck's rising sea philosophy which says that these funny theorems like the nullstellensatz are really just specific cases of some general theorem which is much easier to prove. Next semester we will see this same situation with the Riemann-Roch theorem.

We have the following from (i):

Corollary 6. *Any scheme which is lft over a field is Jacobson.*

Proof. $\operatorname{Spec} k$ is Jacobson. □

We have the following from (iii):

Corollary 7. *If X is lft over $k = \overline{k}$ then $X_{cl} = X(k)$ where $X(k)$ consists of the k-points $(\operatorname{Spec} k \to X)$.*

Proof. Finite algebraic extensions of an algebraically closed field are just the field. □

To prove the theorem we want to reduce to X and Y affine. We will do this by covering Y with affines, and then covering their preimages in X. To see that showing this is sufficient, we notice that these qualities are all local, except we will end up with a Jacobson covering of X which doesn't a priori mean X is Jacobson. So we need the following lemma:

Lemma 1. *X has an open covering of Jacobson spaces iff X is Jacobson.*

Proof. It is certainly true that for X Jacobson, $U \subseteq X$ open implies U is Jacobson. This is clear from $Z \neq \emptyset$ locally closed implies $Z \cap X_{\mathrm{cl}} = \emptyset$.

For the converse, take $\emptyset \neq Z \subseteq X$ locally closed. For some α we have $Z \cap U_\alpha \neq \emptyset$ which is locally closed. Since U_α is Jacobson we can find a closed point $p \in Z \cap U_\alpha$, but this is not necessarily closed in X. But since X can be covered by Jacobson open subsets, then p is closed in X. To see this, we first notice that $\{p\} = \overline{\{p\}} \cap U_\alpha$ which means $\{p\}$ is locally closed, which means p is closed not only in this U_α but in all of the U_α. This means p is a closed point in X. □

Proof of theorem 1. We've reduced the theorem to the case where $X = \operatorname{Spec} B \to Y = \operatorname{Spec} A$ where B is a f.g. A-algebra.

Now we want to reduce the proof further. Another thing to notice, is that this is sort of well-behaved with respect to composition in the sense that if it is true for $f : X \to Y$ and $g : Y \to Z$ then it is clearly true for $g \circ f$.

B f.g. over A just means $B = A[x_1, \cdots, x_n]/J$ for some ideal J. Now we have the following tower:

$$\begin{array}{cc}
A[x_1, \cdots, x_n]/J & \operatorname{Spec} B \\
\uparrow & \downarrow \\
A[x_1, \cdots, x_n] & \mathbb{A}^n_A \\
\uparrow & \downarrow \\
A[x_1, \cdots, x_{n-1}] & \mathbb{A}^{n-1}_A \\
\uparrow & \downarrow \\
\cdots & \cdots \\
\uparrow & \downarrow \\
A[t] & \mathbb{A}^1_A \\
\uparrow & \downarrow \\
A & \operatorname{Spec} A
\end{array}$$

where $\operatorname{Spec} B \to \mathbb{A}^n_A$ is a closed embedding. Another thing to note is that if $X \to Y$ is a closed embedding, this theorem is trivial. Therefore the proof all reduces to $\operatorname{Spec} A[t] = \mathbb{A}^1_A \xrightarrow{p} \operatorname{Spec} A$. It turns out the following is enough:

Claim 1. If $\emptyset \neq Z \subseteq \mathbb{A}^1_A$ is locally closed, then $p(Z)$ contains a non-empty locally closed subset of $\operatorname{Spec} A$.

Remark 2. This is true for any ring A.

If A is Jacobson, then $p(Z)$ contains some closed point $y \in (\operatorname{Spec} A)_{\mathrm{cl}}$. Then the fiber $p^{-1}(y)$ is $\mathbb{A}^1_{k(y)}$ since its natural scheme structure is the fiber product

$$\begin{array}{ccc}
p^{-1}(y) & \longrightarrow & \mathbb{A}^1_A \\
\downarrow & & \downarrow \\
\operatorname{Spec} k(y) & \longrightarrow & A
\end{array}$$

Therefore $Z \cap p^{-1}(y)$ is a locally closed subset. But $\mathbb{A}^1_k$ over a field is Jacobson, which we can check by inspection. This just has a bunch of[1] closed points and the generic point, and the closed subsets are just the empty set, the entire space, and finite sets of closed points.

[1] Note there are always infinitely many closed points of $\mathbb{A}^1_k$ even if k is a finite field.

This means $p^{-1}(y) \cap Z$ contains some $x \in \left(p^{-1}(y)\right)_{\mathrm{cl}}$ so $x \in \left(\mathbb{A}_A^1\right)_{\mathrm{cl}}$.

Now we want to show closed points map to closed points. Let $x \in \left(\mathbb{A}_A^1\right)_{\mathrm{cl}}$, then take $Z = \{z\}$. Then the image of this one point set $\{p(z)\}$ is locally closed in $\operatorname{Spec} A$ which means $p(x)$ is closed.

So this claim gives us that $\mathbb{A}_A^1$ is Jacobson and that closed points map to closed points, which means the field extensions would just be $A[t]$ mod a maximal ideal which is a finite algebraic extension. $\square$