LECTURE 36
MATH 256 A

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It's been a while since we had lecture,[1] so recall that we're trying to get a handle on the relationship between classical varieties and schemes. A classical affine variety is some subset $X \subseteq k^n$ with coordinate ring $\mathcal{O}(X) = k[x_1, \cdots, x_n]/\mathcal{I}(X)$ and then we can consider $Y = \operatorname{Spec}(\mathcal{O}(X))$ which presumably has something to do with X geometrically. There are a few ingredients involved here:

i Sober spaces, which have the property that every irreducible closed subset is the closure of a unique point. Then we can form $\operatorname{Sob}(X)$, and we can send $i : X \to \operatorname{Sob}(X)$ which is a quasi-homeomorphism.

ii Jacobson spaces, which have the property that for all closed Z, Z_{cl} is dense in Z. We saw that we could characterize this in a few different ways. If we take a T_1 space, and we form Sob of these spaces, we get the Jacobson spaces, and we can recover the initial space by taking the closed points of the corresponding Jacobson space.

$$T_1 \text{ spaces} \underset{(-)_{\mathrm{cl}}}{\overset{\mathrm{Sob}}{\rightleftarrows}} \text{Jacobson sober spaces}$$

but on the right we need that the maps are $f : Y \to Y'$ such that $f(Y_{\mathrm{cl}}) \subseteq Y'_{\mathrm{cl}}$. On the left side we have classical varieties over k, and on the right side we get the reduced schemes locally of finite type, and the relevant maps will be morphisms of schemes over k.

iii Morphisms $f : X \to Y$ of schemes which are locally of finite type, which means that for every affine in Y and any affine contained in its preimage in X, the corresponding ring is finitely generated over the other, which we saw is actually a local condition. In retrospect, it's maybe better to define it as:

Definition 1. A morphism $f : X \to Y$ is lft if for every $x \in X$, there exists an open affine neighborhood $x \in U = \operatorname{Spec} B \subseteq X$, and an open neighborhood $f(x) \in V = \operatorname{Spec} A \subseteq Y$ such that $f(U) \subseteq V$, and B is finitely generated over A.

Then it's a theorem that

Theorem 1. *For every $X = \operatorname{Spec} B \subseteq X$, and $V = \operatorname{Spec} A \subseteq Y$ such that $f(U) \subseteq V$, then B is finitely generated over A.*

So it's a well behaved notion. Then we saw that the composition of morphisms lft is also lft, and if the composite is lft, then the first portion

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[1] Because of thanksgiving break and the smoke from forest fires causing class to be cancelled.

is, i.e. if we have

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow^{g} \quad \swarrow & \\ & S & \end{array}$$

then g being lft implies that f is lft.

With all this in hand, we have the following:

Theorem 2. *If $f : X \to Y$ is a morphism locally of finite type, where Y is Jacobson, then*

(i) X is Jacobson
(ii) $f(X_{cl}) \subseteq Y_{cl}$
(iii) If $x \in X_{cl}$, then if we write $f(x) = y$, we have $k(y) \subseteq k(x)$, and this extension is finite algebraic.[2]

Recall the proof of this reduced to $f : \mathbb{A}^1_R = \operatorname{Spec} R[x] \to \operatorname{Spec} R$ and in this case, we had the following:

Lemma 1. *For any ring R, the morphism $\pi : \mathbb{A}^1_R \to \operatorname{Spec} R$ has the property that if $\emptyset \neq Z \subseteq \mathbb{A}^1_R$ is locally closed, then $\pi(Z)$ contains a nonempty locally closed $W \subseteq \operatorname{Spec} R$.*

The piece of commutative algebra that makes this all work is Nakayama's lemma:

Lemma 2 (Nakayama)**.** *For a local ring $(R, \mathfrak{m})$, and a finitely generated R-module M, if $M/\mathfrak{m}M = M \otimes_R K(R) = 0$, then $M = 0$.*

Proof. Since M is fg, let $x_1, \cdots, x_m$ be the generators, so

$$M = \sum_{i=1}^{m} R x_i$$

and by the assumption, we have

$$\mathfrak{m}M = \sum \mathfrak{m} x_i = M \ .$$

Then we can write:

$$x_i = \sum_{j=1}^{n} a_{ij} x_j$$

for $a_{ij} \in M$, which in matrix form is just saying

$$(I - A) \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = 0$$

for $A = \{a_{ij}\}$. Then whatever the determinant is, we have $\det(I - A) \equiv 1 \pmod{m}$, so $\det(I - A) \notin \mathfrak{m}$, which implies that $I - A$ is invertible in $M_n(R)$, so all $x_i = 0$. □

We certainly needed this to be finitely generated, to see this consider the following example:

[2] In particular, if we have a scheme lft over $k = \overline{k}$ then all its closed points have residue field equal to k.

Example 1. Consider $R = k[x]_{(x)} \subseteq k(x)$. This is local where the maximal ideal is generated by x. Then let $M = k(x)$. Notice that $\mathfrak{m}M = xM = M$, so we do have $M/\mathfrak{m}M = 0$, but $M \neq 0$.

We want to use Nakayama's lemma to show the following:

Corollary 1. *Let R be a commutative ring, and B be an R-algebra such that $R \hookrightarrow B$, and B is f.g. as an R-module. Then $\operatorname{Spec} B \to \operatorname{Spec} R$ is surjective.*

Remark 1. The fact that B is f.g. as an R module is saying that $\operatorname{Spec} B \to \operatorname{Spec} R$ is a finite morphism effectively by definition. Then the fact that $R \hookrightarrow B$ is saying that this is dominant since $I = \ker(R \to B)$ will be such that $V(I) = \overline{\operatorname{im}(f)}$ so the image is dense, which means this is a dominant morphism.

Example 2. Note that for $X = \operatorname{Spec} R$ and $X_f = \operatorname{Spec} R\left[f^{-1}\right]$, if f is not a zero divisor, then $R \to R\left[f^{-1}\right]$ has $\ker = 0$, on the other hand the corresponding map $X_f \to X$ is not surjective.

Proof of lemma 1. Without loss of generality we can always shrink Z. In particular, we can assume that Z is a basic open subset X_h for $X \subseteq \mathbb{A}^1_R$ irreducible closed. This means $Z = \operatorname{Spec} B$ where $B = R[x]\left[h(x)^{-1}\right]/Q$ where Q is some prime ideal which means B is an integral domain. Now let $P = Q \cap R$ be the kernel, which is a prime ideal, and then we can replace R with $R/\mathfrak{p}$, and then wlog the map $R \to B$ is injective, so R is an integral domain as well.

Now let $K = K(R)$ be the residue field at the generic point of $\operatorname{Spec} R$, which means if we take $\operatorname{Spec}(K \otimes_R B)$, then the picture is

$$\begin{array}{ccc} & & \operatorname{Spec} B \\ & & \downarrow \\ \operatorname{Spec} k & \longrightarrow & \operatorname{Spec} R \end{array}$$

and the fiber product is the fiber of Z over the generic point of $\operatorname{Spec} R$. So $K \otimes_R B \subseteq K(B)$ is nonzero, so this Z will have points in the fiber over the generic point of $\operatorname{Spec} R$. The other thing to observe is that $\operatorname{Spec}(K \otimes_R B)$ is nonempty, locally closed in $\mathbb{A}^1_K$. Since this is Jacobson, this contains a maximal ideal $\mathfrak{m} = (g(x)) \subseteq k[x]$ for g monic irreducible.

To be continued...

□