LECTURE 37
MATH 256A

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Recall we left off considering the following lemma:

Lemma 1. *For any commutative ring R, consider $\pi : \mathbb{A}^1_R \to \operatorname{Spec}(R)$. For any locally closed subset $\emptyset \neq Z \subseteq \mathbb{A}^1_R$, the image $\pi(Z)$ contains a nonempty locally closed subset of* $\operatorname{Spec} R$.

Proof. WLOG we can make Z smaller, so we can assume $\overline{Z}$ is irreducible. Then it's open in its closure, but we can pass to a basic open subset X_h where X is an irreducible closed subset of $\operatorname{Spec} R$, so $Z = \operatorname{Spec} B$ for the ring $B = R[x]\left[f^{-1}\right]/Q$ for some prime ideal Q. If we intersect $Q \cap R$ we get the image of this in $\operatorname{Spec} R$, and if this is nonzero, we can mod out by this as well and replace R by $R/(Q \cap R)$. So WLOG we can also assume $Q \cap R = 0$, so $R \hookrightarrow B$ and R is an integral domain.

Now consider the fiber of Z over the generic point of $\operatorname{Spec} R$. We can describe this as just being $\operatorname{Spec}(K \otimes_R B)$ where K is the fraction field $K = K(R)$. This fiber is nonempty since this ring is nonzero, so this is a nonempty, locally closed subset of the affine line $\mathbb{A}^1_k$, but we know $\mathbb{A}^1_k$ is Jacobson. Therefore it contains a closed point, which corresponds to an ideal $M \hookleftarrow (g(x)) \subset K[x]$ where g is a monic irreducible polynomial.

We can write $g(x) = f(x)/a$ for $f(x) \in R[x]$ and $0 \neq a \in R$. Now we have that $M \in V(f) \cap Z$, so in particular this is nonempty, so WLOG we might as well replace Z with $V(f) \cap Z$, in other words $f(x) \in Q$ to begin with.

But just as we are allowed to make Z smaller, we can make $\operatorname{Spec} R$ smaller, i.e. we can localize $R\left[a^{-1}\right]$, so we can assume $f(x) \in R[x]$ is monic, and by construction it's irreducible. Now $K[x]/(f(x))$ is a field, and this $h(x)$ is a polynomial in $R[x]$, but it is also a polynomial in $K[x]$, and in particular it is nonzero in $K[x]/(f(x))$, because otherwise f divides h, and since $Z \cap V(h) = \emptyset$, we would have $V(f) \cap Z = \emptyset$ which is a contradiction. Therefore we can invert this as well, so say $h(x)r(x) \equiv 1 \mod f(x)$ in $K[x]$. Now we can write $r(x) = s(x)/b$ where $s(x) \in R[x]$ and $b \in R$, and then this is saying that $h(x)s(x) \equiv b \pmod{f(x)}$ in $R[x]$. But now we can invert b, so WLOG $b^{-1} \in R$, and now we have that $h(x)$ is invertible mod $f(x)$ in $R[x]$, i.e. $h(x)^{-1} \in R[x]/(f(x))$.

Now $f(x) \in Q$, so $B = R[x]/\hat{Q}$ without inverting h. Now this is a quotient of $R[x]$, i.e. $B \twoheadrightarrow R[x]/(f(x))$. So finally, we have $R \hookrightarrow B$, $Z = \operatorname{Spec} B$, and $B \twoheadrightarrow R[x]/(f(x))$, which implies B is a f.g. R-module since $f(x)$ being of degree d, i.e. $f(x) = x^d +$ lower terms, means the set of monomials in $\left\{1, x, \cdots, x^{d-1}\right\}$ generate B as an R-module.

Now by Nakayama, this implies that $\operatorname{Spec} B \twoheadrightarrow \operatorname{Spec} R$, so we are done. ∎

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1. Relationship between schemes lft and varieties

Now we have a nice theorem which tells us that schemes locally of finite type over a Jacobson scheme look like the Sober Jacobson side of this picture.

Theorem 1. *For $f : X \to Y$ lft, and Y Jacobson,*

(i) X is Jacobson
(ii) $f(X_{cl}) \subseteq Y_{cl}$
(iii) If $x \in X_{cl}$, and $y = f(x)$, then $K(y) \hookrightarrow K(x)$ is a finite algebraic extension.

Let $k = \overline{k}$, and let $X \subseteq k^n$ be a classical algebraic variety. Then $\mathcal{O}(X) = k[x_1, \cdots, x_n] / \mathcal{I}(X)$, and we want to consider $Y = \operatorname{Spec} \mathcal{O}(X)$. Note also that we know $\mathcal{O}(X)$ is a finitely generated k-algebra, so $Y \to \operatorname{Spec} k$ is lft, so anything lft over a field is Jacobson. Then for all $y \in Y_{\mathrm{cl}}$, $k(y) = k$. This is saying that for all maximal ideals $\mathfrak{m} \subseteq k[x_1, \cdots, x_n]$, we have $k[\underline{x}]/\mathfrak{m} \simeq k$, which is (the weak form of) Hilbert's Nullstellensatz.

Now notice that the closed points of Y, are exactly the k points, which are exactly X: $Y_{\mathrm{cl}} = Y(k) \cong X$. The correspondence is that a k-algebra homomorphism $\varphi : \mathcal{O}(X) \to k$ corresponds to the point $a \in X$ where $\varphi = \mathrm{ev}_a$. For any subset $I \subseteq \mathcal{O}(X)$, we have $V(I) \subseteq X$, but we also have $V(I) \subseteq Y$, and if we look at the closed points $V(I)_{\mathrm{cl}} \subseteq Y_{\mathrm{cl}}$, then clearly $X \simeq Y_{\mathrm{cl}}$. This shows us that $Y(k) \cong X$ is a homeomorphism.

So this is looking good, Y is a Jacobson sober space, X is T^1 space homeomorphic to closed points of Y, but we have complete equivalence between such spaces, so

$$X = Y_{\mathrm{cl}} \qquad\qquad Y = \operatorname{Sob}(X) \ .$$

In particular, $i : X \to Y$ is a quasi-homeomorphism, so

$$\mathbf{Sh}(X) \underset{i^{-1}}{\overset{i_*}{\rightleftarrows}} \mathbf{Sh}(Y)$$

and these things have the same sheaf theory.

Now we want to see the structure sheaves are actually the same sheaves under this equivalence. On the X side we have $\mathcal{O}_X(U)$. Recall $\mathcal{O}_X(U)$ consists of functions $U \to k$ locally of the form $f = g/h$ on X_h. On the Y side we have $\mathcal{O}_Y$. These things are somehow similar, but it's not obvious they're the same. We defined $\mathcal{O}_Y$ in terms of stalks and germs, but we also had a theorem that $\mathcal{O}_Y(Y_h) = R\left[h^{-1}\right]$. Restriction extends uniquely to a map $R\left[h^{-1}\right] \to \mathcal{O}_X(X_h)$. And we claim, that this is in fact injective. The kernel consists of functions $g/h^n \mapsto 0 \in \mathcal{O}_X(X_h)$. This just means $g \mapsto 0 \in \mathcal{O}_X(X_h)$, in other words $X_h \subseteq V(g)$, but $X_h \coloneqq X \setminus V(h)$, which is saying $X \subseteq V(g) \cup V(h)$, in other words $gh = 0$ everywhere, which means $g = 0 \in R\left[h^{-1}\right]$ so this is injective.

Now we compare stalks at the closed points $x \in X_{\mathrm{cl}}$. On the X side we have $\mathcal{O}_{X,x} \coloneqq \varinjlim \mathcal{O}_X(X_g)$ for $x \in X_h$ and on the Y side we have $\mathcal{O}_{Y,x} = R_{\mathfrak{m}(x)}$. So by construction $\mathcal{O}_{Y,x} \twoheadrightarrow \mathcal{O}_{X,x}$, but it's also injective since everything in $\mathcal{O}_{Y,x}$ which maps to $0 \in \mathcal{O}_{X,x}$ must be 0 in some $R\left[h^{-1}\right]$, possibly smaller. This is the canonical map $i^{-1} : \mathcal{O}_Y \to \mathcal{O}_X$.

So at this point we have $X = Y_{\mathrm{cl}}$ and $Y = \operatorname{Sob}(X)$, but we also have

$$\mathcal{O}_X \underset{i^{-1}}{\overset{i_*}{\rightleftarrows}} \mathcal{O}_Y \ .$$