LECTURE 38
MATH 256A

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1. Affine varieties

Recall for $X \subseteq k^n$ a classical variety over $k = \overline{k}$, and $Y = \operatorname{Spec} \mathcal{O}(X)$, we have $X = Y_{\mathrm{cl}}$, and $Y = \operatorname{Sob}(X)$. In particular $i : X \to Y$ is a quasi-homeomorphism, and the sheaf theory is the same:

$$\mathcal{O}_X \underset{i^{-1}}{\overset{i_*}{\rightleftarrows}} \mathcal{O}_Y \quad .$$

Corollary 1. $\mathcal{O}_X(X) = \mathcal{O}(X)$

This is because we proved this for schemes, so this follows from the above equivalence.

Corollary 2. *Closed $Z \subseteq X$ correspond to radical ideals $I = \sqrt{I} \subset \mathcal{O}(X)$. In particular, $Z = V(I)$. In addition, irreducible subsets correspond to prime ideals.*

We have the fact

Fact 1. $\mathcal{I}(k^n) = 0$*, which means* $\mathcal{O}(k^n) = k[x_1, \cdots, x_n]$.

But even without this, for any field at all we have that $k[x_1, \cdots, x_n]$ is Jacobson. But then we also know that every maximal ideal is the kernel of a k-algebra homomorphism to k since k is algebraically closed. Then we see where the x_i go, and $(x_1 - a_1, \cdots, x_n - a_n) = \ker \mathrm{ev}_a$ for $a \in k^n$. But the intersection of all prime ideals is the 0 ideal, so $\mathcal{I}(k^n) = 0$.

We also know that if we have a reduced f.g. k-algebra, this means $R = k[x_1, \cdots, x_n]/I$ for some radical ideal $I = \sqrt{I}$. Therefore R is the coordinate ring $\mathcal{O}(X)$ for some classical X.

2. Morphisms

2.1. Classical morphisms.

Now we want to think about morphisms. Let X and X' be classical varieties. Then a morphism $f : X \to X'$ is continuous, and gives us a canonical $f^\flat : \mathcal{F}^k_{X'} \to f_* \mathcal{F}^k_X$. But then we want to require that this sends regular functions to regular functions, i.e. $f^\flat(\mathcal{O}_{X'}) \subseteq f_*(\mathcal{O}_X)$.

Recall that morphisms of schemes come with extra structure, whereas in the classical case we get this for free.

Still supposing X and X' are affine, this maps $f^\flat : \mathcal{O}_{X'}(X') \to \mathcal{O}_X(X)$ but in fact this is a k-algebra homomorphism $f^\flat : \mathcal{O}(X') \to \mathcal{O}(X)$, which must correspond

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to a k-morphism $g : Y \to Y'$. Then we have the inclusions $X \hookrightarrow Y$ and $X' \hookrightarrow Y'$ and in fact the following diagram commutes:

$$\begin{array}{ccc} Y & \xrightarrow{\quad g \quad} & Y' \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle i'} \\ X & \xrightarrow{\quad f \quad} & X' \\ \| & & \| \\ Y_{\mathrm{cl}} = Y(k) & & Y'_{\mathrm{cl}} = Y'(k) \end{array}$$

$$\mathrm{ev}_a \longmapsto \mathrm{ev}_{f(a)}$$

This is all reversible, which means the correspondence between classical affine varieties and the corresponding scheme is actually an equivalence of categories. In particular we have the canonical bijection

$$\mathrm{Mor}_{\mathrm{cl}}(X, X') \cong \mathrm{Hom}_{k\text{-}\mathbf{Alg}}(\mathcal{O}(X'), \mathcal{O}(X)) \cong \mathrm{Mor}_{\mathbf{Sch}/k}(Y, Y')$$

The category on the left is classical affine varieties, the middle is k-algebras.

Definition 1. An algebraic scheme over a field k is a scheme locally of finite type over k.

Then the category on the far right consists of reduced affine algebraic schemes over k.

3. Non-affine case

Now let X be a generic classical variety. This means X is covered by affine X_α. In particular we have

$$\begin{array}{ccc} & & X_\alpha \\ & \nearrow & \\ X_\alpha \cap X_\beta & & \\ & \searrow & \\ & & X_\beta \end{array}$$

which might not be affine, but then we can cover this with $X_{\alpha,\beta,\gamma}$ affine, and then we have a massive collection of data which describes X. Then we can turn these all into schemes Y_α, Y_β, and $Y_{\alpha,\beta,\gamma}$. So we get a reduced algebraic scheme over k which is glued together in the same way as X. This is all reversible as well.

The upshot of this, is that now we have a complete equivalence between classical varieties over an algebraically closed field, and reduced algebraic schemes (locally of finite type) over k. Explicitly this is still just given by $X = Y_{\mathrm{cl}} = Y(k)$ and $Y = \mathrm{Sob}(X)$ and the maps on sheaves are the same as well.

This extends to morphisms as well since we cover X', cover the preimages X, specify a bunch of gluing data, and then move the picture to schemes, and all together the square commutes:

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i'} \\ Y & \longrightarrow & Y' \end{array}$$

Corollary 3. *Let $k = \bar{k}$. A k-morphism of reduced algebraic schemes over k is determined by the map of underlying spaces. In fact, it is determined by the underlying map of k-points.*

4. Non-algebraically closed fields

Example 1 (Counter-example)**.** Let $Y = \operatorname{Spec} \mathbb{C}$ over $k = \mathbb{R}$. Note $k \neq \overline{k}$. There is an $\mathbb{R}$-algebra homomorphism which is not the identity, $z \mapsto \bar{z}$, which should give us a nontrivial $\mathbb{R}$-algebra homomorphism $Y \to Y$, but Y only has one point so this is impossible.

In general we can sort of fix it. Let k be generic and Y be an algebraic scheme over k. Then $\operatorname{Spec} \overline{k} \times_{\operatorname{Spec} k} Y = Y_{\overline{k}}$ is algebraic over $\overline{k}$. Y is reduced, and if k is perfect[1] then $Y_{\overline{k}}$ is a reduced algebraic scheme over $\overline{k}$, i.e. a variety. Now any k-morphism $\varphi : Y \to Y'$ gives us a morphism $\varphi_{\bar{k}} : Y_{\bar{k}} \to Y'_{\bar{k}}$ since $\bar{k} \otimes_k R \hookleftarrow R$ and of course $k \hookrightarrow \bar{k}$. In particular, $\varphi_{\bar{k}}$ determines φ.

Example 2. Let's do this in our toy example. This corresponds to taking $Y_{\overline{k}} = \operatorname{Spec}(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C})$ which corresponds to

$$\mathbb{C}[x] / \left(x^2 + 1\right) = \mathbb{C}[x] / (x - i)(x + i)$$

So $Y_{\overline{k}} = \operatorname{Spec} \mathbb{C} \amalg \operatorname{Spec} \mathbb{C}$. Therefore this has two points, and there is a nontrivial map from $Y_{\overline{k}}$ to itself.

[1]Characteristic 0, or if $\operatorname{char} k = p$, closed under taking pth roots. For example $\operatorname{char} k = 0$ and finite fields.