LECTURE 4
MATH 256A

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1. General theory of varieties

The general setup is still a closed subvariety $X = V(F) \subseteq k^n$ where F is some set of polynomials. Let $I = (F)$ be the ideal generated in our polynomial ring $k[x_1, \cdots, x_n]$. Note that every $g \in I$ vanishes on X since it is of the form $\sum a_i f_i$ for $f_i \in F$. Therefore the vanishing locus $V(I) = V(F)$. Of course we have that $V(I) \subseteq V(F)$ trivially since $F \subseteq I$, so this equality is really just saying the opposite containment. In addition to this, since the polynomial ring is Noetherian, every ideal is finitely generated, which means every subvariety of affine space can be defined by finitely many equations.

Example 1. Different ideals can of course define different varieties. Consider the ideals $I = \left(x^2\right) \subseteq k[x]$, $J = (x) \subseteq k[x]$. But in fact, $V(I) = V(J) = \{0\}$.

Now consider

$$\mathcal{I}(X) \coloneqq \{f \in k[x_1, \cdots, x_n] \mid f|_X = 0, X \subseteq V(f)\}$$

First note that tautologically $X \subseteq V(\mathcal{I}(X))$ and $I \subseteq \mathcal{I}(V(I))$. However, $\mathcal{I}(V(I))$ is not equal to I in general.

Example 2. Consider I and J as in the example above. Then $\mathcal{I}(V(J)) = I$.

We do however have the following:

Lemma 1. *$\mathcal{I}V\mathcal{I} = \mathcal{I}$ and $V\mathcal{I}V = V$.*

Proof. We know $V\mathcal{I}(Y) \sup Y$, which means $\mathcal{I}(V\mathcal{I}(Y)) \subseteq \mathcal{I}(Y)$. On the other hand, $\mathcal{I}V(\mathcal{I}(Y)) \supseteq \mathcal{I}(Y)$. Basically the same holds for $V\mathcal{I}V$. ☐

In fact for a general $S \subseteq k^n$, $V(\mathcal{I}(S))$ is the smallest closed subvariety containing S. In a topological sense, X is the closure of S.

Notice that if $f^m \in I$, then $f \in \mathcal{I}(V(I))$, which means $\mathcal{I}(V(I)) \supseteq \sqrt{I}$. In fact this is an equality by Hilbert's Nullstellensatz[1]

Another basic observation we can make is that any intersection of varieties in k^n is a variety. For some varieties $X_\alpha = V(I_\alpha)$ for ideals I_α, we can consider the variety $V\left(\sum_\alpha I_\alpha\right)$ which is clearly the intersection of the X_α. In addition,

$$V(I_1) \cup V(I_2) = V(I_1 I_2) = V(I_1 \cap I_2)$$

Date: August 29, 2018.

[1] This is often times treated as a fundamental result in algebraic geometry, however the analogous result in the theory of schemes is almost tautological, so we will wait until then to "prove" it.

The first equality here is effectively obvious, but the second equality is a bit more subtle. It is clear that $I_1 I_2 \subseteq I_1 \cap I_2 \subseteq I_1, I_2$, but in fact

$$V\left(I_1 I_2\right) \supseteq V\left(I_1 \cap I_2\right) \supseteq V\left(I_1\right) \cup V\left(I_2\right)$$

as desired. Also note that $k^n = V(0)$ and $\emptyset = V((1))$.

These observations tell us that the varieties $X \subseteq k^n$ form the closed subsets of a topology on k^n, called the Zariski topology.

Example 3. Consider the affine line k^1. The whole line is a subvariety, and any finite set of points is a subvariety given by the zero locus of the polynomial with those roots. In fact any one equation cuts you down to a finite set of points, and any additional equation makes you smaller. So the closed subsets are just k^1 and finite subsets. Alternatively, every nonempty open subset is dense, so the affine line cannot be written as the union of two proper closed subsets, so it is in fact irreducible.

Lemma 2. *Let P be a prime ideal. $V(P)$ is irreducible.*

2. Ring of functions on a variety

Let $X \subseteq k^n$ be a variety. Then consider the polynomial ring $k\left[X_1, \cdots, X_n\right]$ and evaluate at the points of X to map

$$k\left[X_1, \cdots, X_n\right] \to \{f : X \to k\} = \mathrm{Map}(X, k)$$

We will write the image as $\mathcal{O}(X)$, the regular functions on X. We can think of this as polynomial functions on X.

Example 4. If X is a single point, $\mathcal{O}(X) = k$. In general, for finite X, $\mathcal{O}(X) \simeq k \times \cdots \times k$.

Since, by definition, $k\left[X_1, \cdots, X_n\right] \twoheadrightarrow \mathcal{O}(X)$, we just need to mod out by the kernel, to find $\mathcal{O}(X)$, i.e.

$$\mathcal{O}(X) \cong k\left[X_1, \cdots, X_n\right] / \mathcal{I}(X)$$

2.1. **Maps between varieties.** Recall that a morphism of varieties $\varphi : X \to Y$ is a map of sets such that the coordinates y_i of $\varphi(x)$ are polynomials in the coordinates x_i of x.

In terms of the rings of regular functions on these varieties, we can regard each coordinate $y_i \in \mathcal{O}(Y)$. By definition $\mathcal{O}(Y)$ consists of functions which are polynomials in these coordinates. Now we can consider the pullback $\varphi^*(y_i) = f_i\left(X_1, \cdots, X_n\right)$ for each i. A priori φ^* is a map from $\mathcal{O}(Y)$ to the collection of all functions $X \to k$, but in fact $\varphi^* : \mathcal{O}(Y) \to \mathcal{O}(X)$ is a k-algebra homomorphism.

$$\begin{tikzcd}
\mathcal{O}(Y) \arrow[rr, "\varphi^*"] \arrow[dr, dashed] & & \mathrm{Map}(X, k) \\
& \mathcal{O}(X) \arrow[ur, hook] &
\end{tikzcd}$$

Conversely, if we have any k-algebra homomorphism, $\alpha : \mathcal{O}(Y) \to \mathcal{O}(X)$, then for each i, $\alpha(y_i) = f_i\left(X_1, \cdots, X_n\right)$ so we have a list of polynomials, which gives us a map $k^n \to k^m$ which sends

$$x \mapsto \left(f_1(x), \cdots, f_n(x)\right)$$

so in the end we get a map $X \to Y$ which is a morphism of varieties by construction.

In conclusion, we get

$$\mathrm{Mor}\left(X, Y\right) \xrightarrow{\sim} \mathrm{Hom}_{k\text{-}\mathbf{Alg}}\left(Y, X\right)$$

where we just map $\varphi \mapsto \varphi^*$.

Example 5. If we map a point $p \hookrightarrow X$, then the evaluation map $\epsilon_p : \mathcal{O}\left(X\right) \to k$ is a k-algebra homomorphism, and conversely, every k-algebra homomorphism from $\mathcal{O}\left(X\right) \to k$ will just be evaluation at a point. So $\mathcal{O}\left(X\right)$ determines X in a concrete sense since it tells us all the points, polynomials on the points, and the whole Zariski topology.

We will do some concrete examples next time.