LECTURE 5
MATH 256A

LECTURES BY: PROFESSOR MARK HAIMAN,
NOTES BY: JACKSON VAN DYKE

1. Rings of functions

Recall last time we were considering a morphism of varieties $\phi : X \to Y$ for $Y \subseteq k^m$ and $X \subseteq k^n$. Then we have the pullback $\phi^* : \mathcal{O}(Y) \to \mathcal{O}(X)$, and in fact we have a bijection:

$$\mathrm{Mor}(X, Y) \xrightarrow{\sim} \mathrm{Hom}_{k\text{-}\mathbf{Alg}}(\mathcal{O}(Y), \mathcal{O}(X))$$

What this is basically saying, is the following:

Proposition 1. *Given a k-algebra homomorphism $\alpha : \mathcal{O}(Y) \to \mathcal{O}(X)$, there is a unique morphism $\phi : X \to Y$ such that $\alpha = \phi^*$.*

To see why this is true we consider the following motivating examples:

Example 1. Consider $Y = k^m$, and $X = \{\mathrm{pt}\}$. Then $\mathcal{O}(Y) = k[Y_1, \cdots, Y_m]$, and $\mathcal{O}(X) = k$. Now to be a k-algebra homomorphism $k[Y_1, \cdots, Y_m] \to k$ the base ring must map to itself, so we just need to specify this to be: $y_i \mapsto \alpha(y_i) = a_i$ and then $Q = (\alpha_1, \cdots, \alpha_m) \in k^m = Y$. Then the pullback is just the evaluation $\mathrm{ev}_Q : f \mapsto f(Q)$.

Example 2. Say $X = \{\mathrm{pt}\}$, and $Y \subseteq k^m$ generic, then $\mathcal{O}(Y) = k[Y]/\mathcal{I}(Y)$. Then we have

$$\begin{array}{ccc} k[X_1, \cdots, X_m] & \twoheadrightarrow & \mathcal{O}(Y) \\ & \overset{\epsilon_Q}{\searrow} & \downarrow \alpha \\ & & k \end{array}$$

for $Q \in Y$.

Now if we want X to also be generic, then say we're given $\alpha : \mathcal{O}(Y) \to \mathcal{O}(X)$, and we want such a ϕ, then we take $P \in X$, which of course has an evaluation map ϵ_P. Then we can compose with α, to get

$$\begin{array}{ccc} \mathcal{O}(Y) & \xrightarrow{\alpha} & \mathcal{O}(X) \\ & \overset{\epsilon_Q}{\searrow} & \downarrow \epsilon_P \\ & & k \end{array}$$

and we can just define $\phi(P) = Q$, such that $\epsilon_Q = \epsilon_P \circ \alpha$. Then for $f \in \mathcal{O}(Y)$,

$$(\phi^* f)(P) = f(\phi(P)) = f(Q) = \epsilon_Q(f) \overset{[1]}{=} \epsilon_Q(\alpha(f)) = \alpha(f)(P)$$

Date: August 31, 2018.

[1] From the diagram

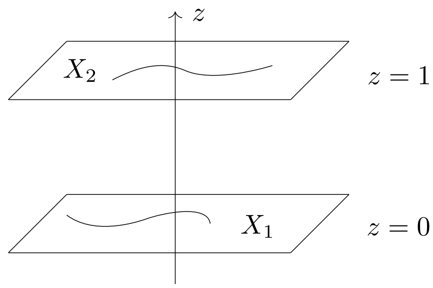

FIGURE 1. If we add an extra dimension to our ambient space, we can rig it such that the X_1 and X_2 are disjoint in this space.

So we have indeed constructed a unique map ϕ such that $\phi^* = \alpha$.

1.1. Examples.

Example 3. If $X = k^n$, then $\mathcal{O}(X) = k[X_1, \cdots, X_n]$. If $X = \{\mathrm{pt}\}$, then $\mathcal{O}(X) = k$.

Example 4. If $X = \emptyset$, $\mathcal{O}(X) = 0$. Note that the zero ring is technically a k-algebra.[2]

If we take any variety Y, there is of course a morphism $\phi : \emptyset \to Y$, and of course there is a unique morphism $\mathcal{O}(Y) \to 0$ so this is consistent. When is there a map $Y \to \emptyset$? Never, unless Y is empty. If such a map did exist, then we would expect a map $0 \to \mathcal{O}(Y)$, but there is no copy of k in 0, and there is one in $\mathcal{O}(Y)$, so for nonempty Y there is also no such k-algebra homomorphism, so this is also consistent.

Example 5. Say $Z \subseteq Y \subseteq k^m$. Then $\mathcal{O}(Z) \supseteq \mathcal{I}(Y)$. Then $\mathcal{O}(Y) \twoheadrightarrow \mathcal{O}(Z)$, since $\mathcal{O}(Y) = k[Y]/\mathcal{I}(Y)$, and similarly for X. The kernel is $\mathcal{I}(Z)$ in $\mathcal{O}(Y)$, so $\mathcal{O}(Z) = \mathcal{O}(Y)/\mathcal{I}(Z)$.

Example 6. If we have two closed varieties, we saw the union is as well, but now let's say we have two varieties X_1 and X_2, then we can consider $X_1 \amalg X_2$. In coordinate land, we want to embed these as disjoint varieties $X_1 \subseteq k^n$ $X_2 \subseteq k^n$, but we don't want to do this, in case they intersect. So let's say they do embed in k^n and X_1 has some equations I_1, and X_2 has some equations I_2. Then we want to embed their disjoint union in k^{n+1}. The picture is as in fig. 1. So the new equations are $z(z-1)$, zI_2, and $(z-1)I_1$, and take ideal I generate by all of these.

Exercise 1. Show that $\mathcal{O}(X_1 \amalg X_2) \cong \mathcal{O}(X_1) \times \mathcal{O}(X_2)$.

Example 7. What is $X \times Y$ in this case? Well if $X = V(I) \subseteq k^n$, and $Y = V(J) \subseteq k^m$, then $V(I, J) \subseteq k^{n+m}$ is, at least set theoretically, $X \times Y$. Algebraically, if we take

$$k[X_1, \cdots, X_n] \otimes_k k[Y_1, \cdots, Y_n] = k[X_1, \cdots, X_n, Y_1, \cdots, Y_n]$$

Now $\mathcal{O}(X)$ is a quotient of this ring by I, and $\mathcal{O}(Y)$ is a quotient of this ring by J, and if tensor these, since tensor products are right-exact, we have

$$k[X_1, \cdots, X_n]/I \otimes_k k[Y_1, \cdots, Y_n]/J = k[X] \otimes k[Y] / (I \otimes (1) + (1) \otimes J)$$

[2] Since a k-algebra is just a ring R along with a ring homomorphism $k \to R$.

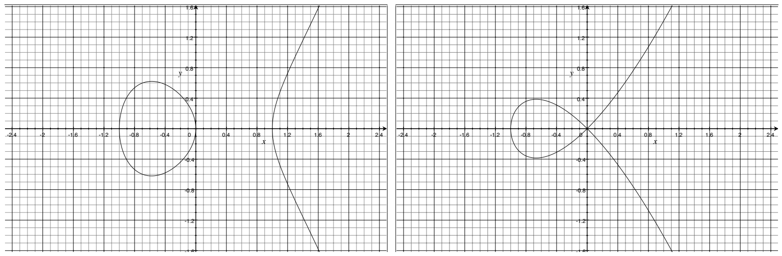

FIGURE 2. (Right) The case of the elliptic curve considered in example 9 for $c \neq 0, 1$. (Left) The case of the elliptic curve form fig. 2 when $c = 0$.

Example 8. Consider an affine plane curve, forgetting about projective space. So let's take something of the form $V(y - f(x))$. This is just a graph of a function. Because this isn't the most general version of such a curve, we can just project to the x-axis. Clearly this projection is a morphism $k^2 \to k^1$. Of course we can also apply this to any subvariety. In the language from before, $Y = V(y)$ is the x-axis. Then this projection will correspond to a ring homomorphism

$$k[x] \to k[x,y]/(y - f(x))$$

where $f(x)$ goes to itself thought of as a function of both x and y, lying in this quotient ring.

$$\begin{array}{ccc} k[x] & \longrightarrow & k[x,y]/(y - f(x)) \\ x & \longmapsto & x \end{array}$$

$$\begin{array}{ccc} k[x] & & k[x,y]/(y - f(x)) \\ x & \longleftarrow\!\shortmid & x \\ f(x) & \longleftarrow\!\shortmid & y \end{array}$$

In general, for any $\phi : X \to Y$, we have the graph $\Gamma(\phi) \subseteq X \times Y$ where we view $X \times Y$ as a variety as in example 7. Then the projection of this graph is an isomorphism.

Example 9. Consider the elliptic curve $y^2 = x(x+1)(x-c)$, this has a graph as in the left figure of fig. 2. We will try to parametrize such a curve. Consider either $c = 0$ or $c = 1$, which looks like the right figure in fig. 2. Take $y^2 = x^2(x+1)$, i.e. set $c = 0$ Then we can consider this on the affine line $k^1 = k[t]$. Now to parametrize this curve we take $x = t^2 - 1$ and $y = t\left(t^2 - 1\right)$. Now to check, we calculate:

$$x^2(x+1) = \left(t^1 - 1\right)^2 t^2 = y^2$$

As expected, we have two values of t such that we hit the origin at those t values. This is $t = \pm 1$. In terms of algebra homomorphisms, we have:

$$k[x, y] / \left(y^2 - x^2 (x+1)\right) \longrightarrow k[t]$$

$$x \longmapsto t^2 - 1$$

$$y \longmapsto t\left(t^2 - 1\right)$$

The way to find this parameterization is as follows. The equation is of degree 3, which means a general line meets it at three points. However, if the line passes through the origin, two roots are already determined. So a general line through the origin meets the curve at one unique additional point. So consider the slope of such a line, and then we can predict that this yields the above parameterization. If we simultaneously set

$$y^2 = x^2 (x+1) \qquad\qquad y = tx$$

we get $t^2 x^2 = x^2 (x+1)$, or $x^2 \left(x + 1 - t^2\right) = 0$.