LECTURE 6
MATH 256A

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1. More elliptic curves

Example 1. In general, elliptic curves are of the form $y^2 =$ some cubic. If it has three roots, we get a nondegenerate curve, and if it has only 1 or 2 roots, we get a sort of degenerate version. Last time we saw $y^2 = x^2 (x+1)$, and that there is a morphism from the affine line to this curve, i.e. a parameterization. Explicitly this was $t \mapsto \left(t^2 - 1, t\left(t^2-1\right)\right)$. $t = \pm 1$ correspond to the two times we cross the origin.

This gives us an algebra morphism

$$k[X,Y] / \left(Y^2 - X^2 (X+1)\right) \to k[t]$$

where $x \mapsto t^2 - 1$ and $y \mapsto t\left(t^2 - 1\right)$. A priori this ideal is only inside the ideal of this curve, but one way to check that this is the full ideal of the curve, is to check that this is the full kernel of the morphism $k[X,Y] \to k[t]$. In general ideal theory is somehow hard, and this is an example of how to simplify such questions.

Example 2. Now consider $y^2 = x^3$. This has a cusp at the origin as in fig. 1. Again we consider the lines $y = tx$, and then $t^2x^2 - x^3 = 0$, so $x^2\left(t^2 - x\right) = 0$, so

Date: September 5, 2018.

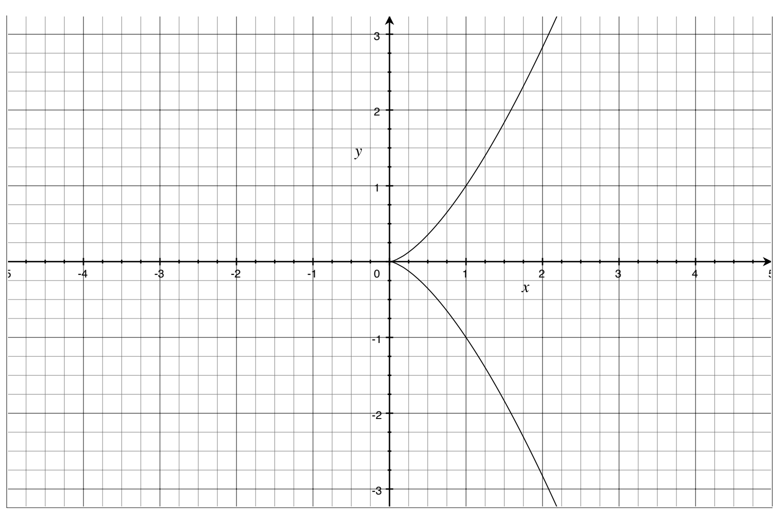

Figure 1. The curve $y^2 = x^3$.

$x = t^2$ and $y = t^3$ is a parameterization.

Now assuming the ideal of the curve is $\left(y^2 - x^3\right)$, we have a morphism

$$\mathcal{O}(C) = k[x,y] / \left(y^2 - x^3\right) \to k[t]$$

where $x \mapsto t^2$, and $y \mapsto t^3$. The image of this map is $k\left[t^2, t^3\right] \subseteq k[t]$, which is spanned by $\{t^m \mid m \neq 1\}$. This shows us that this cannot be an isomorphism of algebraic varieties, since it is not an isomorphism of algebras. Nonetheless, this map is a bijection onto its image, and a homeomorphism with respect to the Zariski topology.

We can explicitly see that this is the full ideal of the curve in the following way. Consider the ring $k[x,y] / \left(y^2 - x^3\right)$. We know the image in $k[t]$ is a nice subring, so if we can find a nice basis of $k[x,y] / \left(y^2 - x^3\right)$ such that these basis elements map to the basis of $k[t]$, then this is a bijection of vector spaces onto its image, so it is certainly injective, which means this ideal is the full kernel of the map as desired. This ring is spanned by $\{1, y\} \cdot \{1, x, \cdots\}$, and the map brings $x^m \to t^{2m}$, and $yx^m \mapsto t^{2m+3}$ as desired.

Example 3. Let $X \subseteq k^n$ be any classical affine variety, write $I = \mathcal{I}(X)$ so $\mathcal{O}(X) = k[\underline{x}]/I$. Now consider $f \in \mathcal{O}(X)$ and write $\mathcal{U} = X \setminus V(f)$. Since $V(f)$ is closed, $\mathcal{U}$ is open in X with respect to the Zariski topology.

If this was an arbitrary open set, then we can express it as a union of these particular ones:

$$X \setminus V(I) = X \setminus \bigcap_{f \in I} V(f) = \bigcup_{f \in I} (X \setminus V(f))$$

so these form a basis.

A priori this isn't a variety, but we will identify it as a variety. Define a new variety X_f and morphism $i : X_f \to X$ such that i is an injective and open morphism. We should be cautious, since we can have a homeomorphism of varieties that isn't an isomorphism. In particular, we take $X_f \subseteq k^{n+1}$ with the same coordinates as before, except now with a new coordinate z. The equations will be

$$X_f = V\left(\{h(\underline{x}) \mid h \in I\} \cup \{z f(x) - 1\}\right)$$

$$\begin{array}{ccccc} X_f & \subseteq & k^{n+1} & \qquad & (\underline{x}, z) \\ \downarrow & & \downarrow & & \downarrow \\ X & \subseteq & k^n & & (\underline{x}) \end{array}$$

The image of this map is $\mathcal{U}$, so this is injective, and in fact $X_f \xrightarrow{\simeq} \mathcal{U}$.

Let

$$J \coloneqq I \cdot k[\underline{x}, z] + (zf - 1)$$

so $X_f = V(J)$. The above map gives us an algebra homomorphism

$$\begin{array}{ccc} \mathcal{O}(X) & \xrightarrow{i^*} & \mathcal{O}(X_f) \\ \| & & \uparrow \\ k[\underline{x}]/I & \longrightarrow & k[\underline{x}, z]/J \end{array}$$

$$x_i \longmapsto x_i$$

Now $k[\underline{x}, z] = \mathcal{O}(X)[z] / (zf - 1)$ so this is just adjoining the inverse of z, so this surjection is equality.

In any case, for now we just agree that $\mathcal{U} = X \setminus V(f)$ is an affine variety via $X_f \cong \mathcal{U}$, i.e. $\mathcal{O}(\mathcal{U}) = \mathcal{O}(X)\left[f^{-1}\right]$.

2. Sheaves

Let X be a topological space.

Definition 1. A pre-sheaf $\mathcal{S}$ (of sets[1]) on X assigns to each open set $U \subseteq X$ a set $\mathcal{S}(U)$, with maps $\rho_{UV} : \mathcal{S}(U) \to \mathcal{S}(V)$ for $V \subseteq U$ which satisfy:

(1) If $W \subseteq V \subseteq U$, then $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$.
(2) ρ_{UU} is the identity.

In other words, if we regard the open sets of X as a category with morphisms given by inclusions, then $\mathcal{S}$ is a contravariant functor from this category to **Set**.

[1] We can have sheaves of abelian groups etc.