LECTURE 7
MATH 256A

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1. Motivation

Besides algebraic geometry, there are other flavors such as the study of topological manifolds, smooth manifolds over $\mathbb{R}$, analytic manifolds over $\mathbb{C}$, algebraic varieties etc. One thing that they all have in common, is they have an underlying topological space. In algebraic geometry, we want the underlying topological space to have the Zariski topology, which is effectively as non-Hausdorff as possible. For topological manifolds, the topology effectively tells the whole story, but for the other flavors, you need to know more than just the topology, because you need to know what a smooth coordinate system looks like.

If you've seen a bit of differential geometry, you know that a manifold is supposed to locally have coordinates, usually presented as an atlas of charts. Then if you want it to be smooth, you require the overlaps to be smooth. The same story happens for complex analytic manifolds. One problem with this, is that if you want an invariant definition, then you have to say something about this family being maximal, which makes the atlas hard to directly work with. But there's a better way to do this. Instead of specifying the coordinate charts, you can specify the "smooth" functions and then you can recover the coordinate charts from this. So it's somehow better to think of these objects as topological spaces equipped with a notion of "good" functions. This is where the notion of a sheaf of functions comes in.

2. Sheaves

2.1. **Presheaves.** Let X be a topological space.

Definition 1. A *presheaf* $\mathcal{S}$ (of sets[1]) on X assigns to each open set $U \subseteq X$ a set $\mathcal{S}(U)$, with maps $\rho_{UV} : \mathcal{S}(U) \to \mathcal{S}(V)$ for $V \subseteq U$ which satisfy:

(1) If $W \subseteq V \subseteq U$, then $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$.
(2) ρ_{UU} is the identity.

In other words, if we regard the open sets of X as a category $\mathbf{Open}(X)$ with morphisms $A \to B$ iff $B \subseteq A$. Then $\mathcal{S}$ is a functor from this category to **Set**.[2]

Let's say we have $\mathcal{A}$ a presheaf of rings on X, and $\mathcal{M}$ a presheaf of abelian groups. Then for every $U \subseteq X$, we have $\mathcal{A}(U) \times \mathcal{M}(U)$ making $\mathcal{M}(U)$ an $\mathcal{A}(U)$

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[1] We can have sheaves of abelian groups etc.

[2] Or **Grp** or **Ring** etc.

module. Now for $V \subseteq U$, we also have $\mathcal{A}(V)$ and $\mathcal{M}(V)$ and

$$\begin{array}{cc} \mathcal{A}(U) & \mathcal{M}(U) \\ \downarrow & \downarrow \\ \mathcal{A}(V) & \mathcal{M}(V) \end{array}$$

I.e. since we have a morphism $\mathcal{A}(U) \to \mathcal{A}(V)$, we can view $\mathcal{M}(V)$ as a $\mathcal{A}(U)$-module, and $\mathcal{M}(U)$ is already a $\mathcal{A}(U)$-module, so now we just require that the morphism on the right is a $\mathcal{A}(U)$-module homomorphism and then we say that $\mathcal{M}$ is a presheaf of modules for the presheaf of rings $\mathcal{A}$.

Example 1. For any X and any set k, $F = \mathrm{Fun}(X, k)$ is a presheaf where

$$F(U) = \{f : U \to k\}$$

and then $\rho_{UV}(f) = f|_V$. Note that if k is a ring, then $\mathrm{Fun}(X, k)$ is a presheaf of rings rather than a presheaf of sets.

Example 2. Given a presheaf $\mathcal{S}$, for each U, if we have subsets $\mathcal{T}(U) \subseteq \mathcal{S}(U)$ such that $\rho_{UV}(\mathcal{T}(U)) \subseteq \mathcal{T}(V)$, then $\mathcal{T}$ is a presheaf, and we say it is a sub-presheaf.

Example 3. Sections of any sort of bundle form a presheaf since we can just take any section and restrict to open sets.

2.2. **Sheaves.** The idea behind the definition of a sheaf is to axiomatize the idea that the things inside a sheaf are local, which is not the case for a generic presheaf.

Definition 2. A presheaf is a sheaf iff: given an open set U, an open covering

$$U = \bigcup_\alpha U_\alpha$$

and $s_\alpha \in \mathcal{S}(U_\alpha)$ for all α, such that for all α, β

$$\rho_{U_\alpha, U_\alpha \cap U_\beta} s_\alpha = \rho_{U_\beta, U_\alpha \cap U_\beta} s_\beta$$

then there exists a unique $s \in S(U)$ such that $s_\alpha = \rho_{U, U_\alpha} s$ for all α.

Example 4. The presheaf $\mathrm{Fun}(X, k)$ from above is actually a sheaf.

Warning 1. If $\mathcal{S}$ is a sheaf, a sub-presheaf might not be a sheaf.