LECTURE 8
MATH 256A

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1. Sheaves

Recall the following definition from last time:

Definition 1. A presheaf is a sheaf iff: given an open set U, an open covering

$$U = \bigcup_\alpha U_\alpha$$

and $s_\alpha \in \mathcal{S}(U_\alpha)$ for all α, such that for all α, β

$$\rho_{U_\alpha, U_\alpha \cap U_\beta} s_\alpha = \rho_{U_\beta, U_\alpha \cap U_\beta} s_\beta$$

then there exists a unique $s \in S(U)$ such that $s_\alpha = \rho_{U,U_\alpha} s$ for all α.

One might complain that this uses the fact that sets have elements, and therefore cannot be adapted to a definition for a sheaf taking values in some category where the objects don't have underlying sets. As it turns out, the definition is equivalent to the map $\prod \rho_{U,U_\alpha}$ being the equalizer of the maps $\prod \rho_{U_\alpha, U_\alpha \cap U_\beta}$ and $\prod \rho_{U_\beta, U_\alpha \cap U_\beta}$ as in the diagram:

$$\mathcal{S}(U) \xrightarrow{\prod \rho_{U,U_\alpha}} \prod_\alpha S(U_\alpha) \underset{\prod \rho_{U_\beta, U_\alpha \cap U_\beta}}{\overset{\prod \rho_{U_\alpha, U_\alpha \cap U_\beta}}{\rightrightarrows}} \prod_{\alpha,\beta} \mathcal{S}(U_\alpha \cap U_\beta)$$

So all we require here is that the category has products. Note that this map clearly always equalizes the two arrows on the right, but being the equalizer means it's universal.

Recall that a sub-presheaf of a sheaf is not always a sheaf. But we do have a condition which forces this to be the case. Given an open cover $\{U_\alpha\}$ of U, and $\mathcal{T} \subseteq \mathcal{S}$ a sub-presheaf of a sheaf $\mathcal{S}$, and compatible $t_\alpha \in \mathcal{T}(U_\alpha)$, then since S is a sheaf, there is a unique $s \in \mathcal{S}(U)$ which restricts to give the t_α so all we require is that this $s \in \mathcal{T}(U)$ as well.

All this is really saying is given a section of $s \in \mathcal{S}(U)$ such that $s|_{U_\alpha} = \mathcal{T}(U_\alpha)$ for all α, then $s \in \mathcal{T}(U)$ as well. The motto here is "being in T is a local condition."

Example 1. Consider the sheaf of all function $\mathrm{Fun}(X, k)$. This is always a sheaf, but now consider any sub-presheaf of this. This is automatically a sheaf whenever it is defined by some sort of local condition. For example, consider the collection of continuous functions $\mathrm{Cts}(X, K) \subseteq \mathrm{Fun}(X, K)$ for any other topological space K.

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Since continuity is a local condition, this is a sheaf. Another example is $\mathcal{C}^\infty(M) \subseteq \operatorname{Fun}(M, \mathbb{R})$ for some smooth manifold M. This is also a sheaf, because this is just saying it has infinitely many derivatives at each point.

2. Classical varieties

Let X be an affine variety over some algebraically closed field k. Recall that open sets of the form $X_g = X \setminus V(g)$ for any $g \in \mathcal{O}(X)$ satisfy the following:

(1) They form a base for the Zariski topology on X.
(2) They are varieties with coordinate ring:
$$\mathcal{O}(X_g) = \mathcal{O}(X)\left[g^{-1}\right]$$
Recall the inclusion $X_g \hookrightarrow X$ corresponds to the ring homomorphism $\mathcal{O}(X) \to \mathcal{O}(X)\left[g^{-1}\right]$.

Example 2. Note that $\mathcal{O}(X)\left[g^{-1}\right]$ is not necessarily larger than $\mathcal{O}(X)$. For example, X might be the union of the x and y axes, defined by the polynomial xy. If $g = x$, then inverting it means $y = 0$ as well.

But if it does happen that g doesn't vanish on any irreducible component of X, then this is in fact a "bigger" ring.

Now we want to define a sheaf of functions on such an X.

Definition 2. Let $U \subseteq X$ be open, then the sheaf of functions $\mathcal{O}_X \subseteq \operatorname{Fun}(X, k)$ is defined by $h \in \mathcal{O}_X(U)$ iff every $x \in U$ has a neighborhood of the form X_g such that $h|_{X_g} \in \mathcal{O}(X_g)$, i.e. $h = f/g^m$ where $f \in \mathcal{O}(X)$.

WLOG we can just insist that $h = f/g$ since $X_g = X_{g^m}$.

Note that this is a local condition, so indeed this is a sub-sheaf of the sheaf of all functions. The following will be proven later, since it will be a corollary of a theorem which we will prove when we get to schemes.

Theorem 1. *Let X be an affine variety, then $\mathcal{O}_X(X) = \mathcal{O}(X)$.*

Corollary 1. $\mathcal{O}_X(X_g) = \mathcal{O}(X)\left[g^{-1}\right]$

So now we know something about affine varieties, but what about generic classical varieties?

Definition 3. A classical variety[1] over an algebraically closed field k is a topological space X with a sheaf $\mathcal{O}_X \subseteq \operatorname{Fun}(X, k)$ such that X can be covered by open sets U for which $(U, \mathcal{O}_X|_U)$ is isomorphic to to some affine variety $(Y, \mathcal{O}_Y)$.

Remark 1. There's something funny here because we haven't really specified what a homomorphism of these objects are. But it's sort of a meta-theorem that you always know what an isomorphism is, since it just preserves all structure.

Lemma 1. *Any open subset $U \subseteq X$ of a classical variety is a classical variety.*

This is already an "improvement" on affine varieties, since affine varieties have plenty of subsets which are not affine.

This statement is not obvious. Of course we can cover X with affine varieties, but why can we cover U? This cant just be intersecting this cover with U, since these intersections might not be affine. For instance, if X is affine itself, maybe

[1] In the sense of Serre.

we just cover that X itself, however there are plenty of non-affine subsets of affine varieties. However the point is, there are plenty of affine subsets of affine varieties, in particular the affine varieties form a base for the topology, we can in fact cover U. For any point in U, we have an affine subset of X which contains this point, and is contained in U, so the union of these covers U.

Example 3. As an example of a classical variety which is not affine, consider $X = k^2$, and $U = X \setminus \{0\}$. For $X_x = V(x)$ and $X_y = V(y)$, we have $U = X_x \cup X_y$ so this is an explicit affine cover. Now what is $\mathcal{O}_U(U)$? We know $\mathcal{O}(X_x) = k\left[x^{\pm 1}, y\right]$ and $\mathcal{O}(X_y) = k\left[x, y^{\pm 1}\right]$ both embed in $\mathcal{O}(X_{xy}) = k\left[x^{\pm 1}, y^{\pm 1}\right]$, where we write $X_x \cap X_y =: X_{xy}$. This means

$$\mathcal{O}_U(U) = k[x, y]$$

But this means U cannot be affine, since if it was, $\mathcal{O}_U(U) = \mathcal{O}(U)$ would be isomorphic to $\mathcal{O}(X)$, but the spaces are not isomorphic themselves.

Example 4. Consider two copies of the affine line, and glue them along the embedding $k^\times \hookrightarrow k$. This is the affine line with two copies of the origin. This thing is a somewhat pathological example, since it is not Hausdorff, but (for now) this is a perfectly fine classical variety, since it is covered by the two affine varieties k.