LECTURE 9
MATH 256A

LECTURES BY: PROFESSOR MARK HAIMAN
NOTES BY: JACKSON VAN DYKE

1. Projective varieties

Now we can formally define projective space as a classical variety. We still won't spend too much time on this, since we will redefine it again in the context of schemes.

The n-dimensional classical projective space $\mathbb{P}^n$ over an algebraically closed field k, is the set of lines $k(x_1, \cdots, x_n) \subset k^{n+1}$ through the origin in k^{n+1}. We write such a line as $(x_0 : \cdots : x_n) \in \mathbb{P}^n$. So we want to show this is a topological space, define a sheaf of functions on it, and show that it is indeed a variety.

Recall that if we have a homogeneous polynomial $f(x_0, \cdots, x_n)$ of degree d this just means $f(t\underline{x}) = t^d f(\underline{x})$. So whether or not f is zero is independent of multiplication by a scalar, i.e. $V(f) \subseteq \mathbb{P}^n$ makes sense.

The polynomial ring $R = k[x_0, \cdots, x_n]$ is a graded ring, where we let R_d consist of homogeneous polynomials of degree d and then

$$R = \bigoplus_{d>0} R_d$$

Whenever we have an ideal $I \subseteq R$, we say I is graded if

$$I = \sum_d (I \cap R_d)$$

Of course in this case,

$$I = \bigoplus_d (I \cap R_d) = \bigoplus_d I_d$$

and

$$R/I = \bigoplus_d R_d / I_d$$

Now we have the following lemma:

Lemma 1. *I is graded iff for all $f \in I$, the homogeneous components of f are in I, iff I can be generated by homogeneous elements.*

This is good, since it means it makes sense to write $V(I) \subseteq \mathbb{P}^n$ for graded ideals I.

Note that $m = (X_0, \cdots, X_m)$ is the unique homogeneous maximal ideal, and $V(m) = \emptyset$, so there is not a one-to-one correspondence between homogeneous radical ideals and subvarieties. However all other homogeneous maximal ideals are

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contained in this one, so the actual correspondence is between homogeneous radical ideals contained in m and subvarieties.

Since $V(IJ) = V(I \cap J) = V(I) \cup V(J)$ works in projective space, and we clearly get the whole space and $\emptyset$, we can take the topology to be such that closed subsets are the subvarieties.

Now what defines a regular function on open subsets of this thing? We want to say this by finding a local condition. Let $f(x)$ be a homogeneous rational function of degree 0. Being rational means it's a quotient of two polynomials, $g(x)/h(x)$, and being of degree 0 means it is invariant under rescaling the argument. It is also true that if we write this in lowest terms, that $g(x)$ and $h(x)$ are homogeneous too. In any case, f makes sense as a function $W_h \to k$ where

$$W_h = \mathbb{P}^n \setminus V(h)$$

This motivates us to define $\mathcal{O}_{\mathbb{P}^n} \subseteq \operatorname{Fun}(\mathbb{P}^n, k)$ to consist of functions locally of this form.

Now if we write:

$$W_i = W_{x_i} = \mathbb{P}^n \setminus V(x_i) \qquad\qquad \mathbb{P}^n = \bigcup_i W_i$$

we claim that $W_i \cong k^n$. By symmetry, it is enough to show this for W_0. If $x \in W_0$, since $x_0 \neq 0$, x has a unique representative $(1 : x_1 : \cdots : x_n)$. So this gives a bijection between $W_0 = k^n$.

Any closed subset of W_0 is some $V(f_1, \cdots)$ which is just the intersection of the $V(f_i)$, so we just consider $V(f)$. f is a homogeneous polynomial in $x_0, \cdots, x_n$ restricted to W_0. On W_0 this is just $f(1, x_1, \cdots, x_n)$, so whatever the vanishing locus of f is, intersected with W_0 will just be the vanishing locus of this new polynomial. Therefore closed in W_0 implies closed in k^n.

Now suppose have a closed subset of k^n, so $V(g)$ for some $g(x_1, \cdots, x_n)$, then we want to show this is the locus where some homogeneous polynomial in projective coordinates vanishes, i.e. we want $f(1, x_1, \cdots, x_n)$ such that $V(g) = W_0 \cap V(f)$. To do this we homogenize the polynomial. Then setting this variable to 1 we get the same polynomial

Example 1. For example, if we have $g(x, y, z) = xy - z$, we set $f(w, x, y, z) = xy - wz$.

This shows that these have the same topology. Now we have to show they have the same sheaf of regular functions, but this is just the same game. If we have a rational function on W_0, this is locally

$$\frac{g(x_0, \cdots, x_n)}{h(x_1, \cdots, x_n)}$$

where g, h are homogeneous polynomials of the same degree, but on $X_h \cap X_0$, this is just

$$\frac{g(1, x_1, \cdots, x_n)}{h(1, x_1, \cdots, x_n)}$$

The other direction is a bit harder. We have to show any rational function can be homogenized. If we have

$$\frac{g(x_1, \cdots, x_n)}{h(x_1, \cdots, x_n)}$$

we can bump g up to a homogeneous polynomial $\hat{g}(1, x_1, \cdots, x_n)$ and similarly for h to get $\hat{h}(1, x_1, \cdots, x_n)$ and in particular, we can get these to have the same degree, so we are finished.

So we have shown any projective variety $Z = V(I) \subseteq \mathbb{P}^n$ is a classical variety by affines $Z \cap W_i$. In particular, any open $U \subseteq Z$ is a variety. This is called a quasi-projective.

2. Elliptic curves

We revisit elliptic curves now that we have a bit firmer foundation. Consider a curve

$$C = V\left(y^2 - (x-a)(x-b)(x-c)\right) \subseteq \mathbb{P}^2$$

Then to homogenize, we write:

$$C = V\left(zy^2 - (x-az)(x-bz)(x-cz)\right) \subseteq \mathbb{P}^2$$

Note that $V(z)$ is the line $\mathbb{P}^1$ at ∞, written $(0 : x : y)$. So setting $z = 0$ in the above we get $V\left(x^3\right) = \{(0 : 0 : 1)\}$.