LECTURE 1
MATH 256B

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This semester we will cover a bit more foundations, such as the Proj construction, which allows us to view projective varieties without this sort of ad hoc covering argument. Then we will do some actual geometry such as dimension theory, smoothness, singularities, etc. We also want to talk about sheaf cohomology and some applications.

1. Proj CONSTRUCTION

1.1. **Classical motivation.** Recall that as a set, $\mathbb{P}^n$ is the collection of lines through the origin in $\mathbb{A}^{n+1}$. We saw this was a variety by covering $\mathbb{P}^n$ with copies of $\mathbb{A}^n$. What about closed subvarieties? Well if we take some homogeneous polynomial $f(x)$ of degree d, i.e. $f(tx) = t^d f(x)$, then $V(f)$ is some union of lines through the origin in $\mathbb{A}^{n+1}$. In other words, $V(f)$ is some sort of cone-like closed subvariety.

Recall irreducible closed subvarieties correspond to prime ideals $P \subseteq k[x_0, \cdots, x_n]$ which are invariant under $k^\times \circlearrowright k^n$. This invariance will be equivalent to P being a graded ideal. Then these primes should correspond to the points of projective space as a scheme, with the exception of the maximal ideal $(x_0, \cdots, x_n)$. The closed variety corresponding to this ideal is just the origin. So really we're just interested in the invariant prime ideals which aren't fixed.

1.2. **Graded rings and ideals.** So we want to start with any graded ring, and we want to put a scheme structure on the set of graded prime ideals, where we exclude ones which contain the whole positive degree part.

Definition 1. A ring R is called $\mathbb{Z}$-graded iff as an abelian group it can be written

$$R = \bigoplus_{n \in \mathbb{Z}} R_n \ .$$

In addition, we need the ring structure to respect this, so we require:

$$R_m \cdot R_n \subseteq R_{m+n}$$

Remark 1. $\mathbb{N}$-graded rings are defined in the same way.

Example 1. For $R = A[x_1, \cdots, x_n]$ we have that R_n consists of homogeneous polynomials of degree n.

Definition 2. Let R be a graded ring, and $I \subseteq R$ an ideal. We say I is graded iff

$$I = \sum_n I_n$$

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where $I_n \subseteq R_n$, so $I = \bigoplus I_n$ as well.

Alternatively, we can say the following. We already know that for any $f \in R$, since R is a direct sum of the R_n, f can be uniquely written as the sum[1]

$$f = \sum_n f_n$$

for $f_n \in R_n$. Then saying I is graded is just saying that if $f \in I$, then $f_n \in I$ for all n.

Lemma 1. *An ideal is graded iff it is generated by its homogeneous elements.*

Proof. Write $I = (g_\alpha)$, for some collection of ring elements $g_\alpha \in R_{n_\alpha}$. An arbitrary element of I looks like the finite sum $\sum r_\alpha g_\alpha$. In particular, the degree n part is $\sum_\alpha (r_\alpha)_{n-n_\alpha} g_\alpha$. ☐

Note that we want this notion of a graded ideal, because if R and I are both graded, then

$$R/I = \bigoplus_n R_n/I_n$$

is graded as well.

Remark 2. We can define graded modules over a graded ring in the same way as graded rings, and graded submodules the same way as graded ideals.

1.3. **Group schemes.** Notice that $k^\times = \operatorname{Spec} k\left[t, t^{-1}\right]$ is actually an algebraic group since the multiplication map $k^\times \times k^\times \to k^\times$ and the inverse map $k^\times \to k^\times$ are morphisms of algebraic varieties. In particular:

$$\begin{tikzcd} k^\times \times k^\times \arrow[r] & k^\times & k\left[t^{\pm 1}, u^{\pm 1}\right] & k\left[s^{\pm 1}\right] \arrow[l] \\ (t,u) \arrow[r, mapsto] & tu & tu & s \arrow[l, mapsto] \end{tikzcd}$$

and similarly

$$\begin{tikzcd} k^\times \arrow[r] & k^\times & k\left[t^{\pm 1}\right] & k\left[t^{\pm 1}\right] \arrow[l] \\ t \arrow[r, mapsto] & t^{-1} & t^{-1} & t \arrow[l, mapsto] \end{tikzcd}$$

Notice that this doesn't use that $k = \overline{k}$ or even that it's a field, so we can generally write that

$$\operatorname{Spec} R\left[t^{\pm 1}\right] \to \operatorname{Spec} R$$

is a group scheme $\mathbb{G}_{m,R}$ over R. The group axioms are then satisfied as commutative diagrams. For example multiplication in a group being associative is equivalent to the following diagram commuting:

$$\begin{tikzcd} G \times G \times G \arrow[r, "\mathrm{id} \times m"] \arrow[d, "m \times \mathrm{id}"] & G \times G \arrow[d, "m"] \\ G \times G \arrow[r, "m"] & G \end{tikzcd}$$

Note that these are not groups. For example, if R is an integral domain, so is $R\left[t^{\pm 1}\right]$, which means this has a generic point, and $\mathrm{pt} \times \mathrm{pt} = \mathrm{pt}$, so it is the identity,

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[1] Note that all but finitely many of the f_n are zero.

but this can't be. What does turn out to be a group is the functor. The functor $\underline{\underline{G}}\left(X\right)$ consists of the R morphisms $X \to G$, and then $\underline{\underline{G}}$ is a group functor.

The sense in which the group scheme has to do with the invertible elements is the following. If we have an R-algebra A, then a map $\operatorname{Spec} A \to \operatorname{Spec} R\left[t^{\pm 1}\right]$ corresponds to a map $R\left[t^{\pm 1}\right] \to A$, and the functor $\underline{\underline{\mathbb{G}_{m,R}}}\left(\operatorname{Spec} A\right) = A^{\times}$ and $\underline{\underline{\mathbb{G}_{m,R}}}\left(X\right) = \mathcal{O}_X\left(X\right)^{\times}$.

Remark 3. Note that $\mathbb{G}_{m,\mathbb{Z}} = \operatorname{Spec} \mathbb{Z}\left[t^{\pm 1}\right]$ somehow knows everything since every scheme is a scheme over $\mathbb{Z}$.

1.4. **The action of $\mathbb{G}_m$ on a scheme.** An action of $\mathbb{G}_m$ on a scheme $X = \operatorname{Spec} R$ is a map $\alpha : \mathbb{G}_m \times_{\mathbb{Z}} X \to X$ such that $\mathbf{1}_G \cdot X \to X$, and $g\left(hx\right) = \left(gh\right)x$. In particular, the identity element is $\mathbf{1}_G : \operatorname{Spec} \mathbb{Z} \to \mathbb{G}_m$, and the statement that it is the identity is that the following diagram commutes:

$$\begin{tikzcd}
X = \operatorname{Spec} \mathbb{Z} \times X \arrow[r, "\mathbf{1}_G \times \mathrm{id}_X"] \arrow[rr, bend right, "\mathrm{id}_X"'] & \mathbb{G}_m \times X \arrow[r, "\alpha"] & X
\end{tikzcd} .$$

In other words, the composition $\alpha \circ \mathbf{1}_{\mathbb{G}} \times \mathrm{id}_X$ is the identity on X. Similarly the associativity condition is equivalent to the following diagram commuting:

$$\begin{tikzcd}
\mathbb{G}_m \times \mathbb{G}_m \times X \arrow[r, "\mathrm{id} \times \alpha"] \arrow[d, "m \times \mathrm{id}"] & \mathbb{G}_m \times X \arrow[d, "\alpha"] \\
\mathbb{G}_m \times X \arrow[r, "\alpha"] & X
\end{tikzcd} .$$

In particular, an action of $\mathbb{G}_m$ on $\operatorname{Spec} R$ is given by a ring homomorphism $R \to R\left[t^{\pm 1}\right]$. Any $f \in R$ maps to a finite sum

$$f \mapsto \sum_{n \in ZZ} t^n f_n$$

and then the fact that this is an action just means

$$\left(f_n\right)_m = \begin{cases} 0 & m \neq n \\ f_n & m = n \end{cases}$$

and the identity axiom says that

$$\sum f_n = f \ .$$

This ring homomorphism gives us a bunch of projections, and in particular a direct sum decomposition of R:

$$R = \bigoplus R_n$$

where $R_n = \{f \mid f_n = r\}$. Note that in the case of the polynomial ring, this is exactly the homogeneous elements of degree n. One can check that this is indeed a grading, i.e. the ring structure respects this decomposition. All of this is to say:

($\mathbb{G}_m$ actions on $\operatorname{Spec} R$) = ($\mathbb{Z}$-gradings of R)

1.5. **The irrelevant ideal.** Let R be $\mathbb{N}$-graded, and let $P \subseteq R$ be a graded prime ideal. Note that the graded ideals are now just ideals invariant under the $\mathbb{G}_m$ action. But now we want to exclude the $\mathbb{G}_m$ fixed ideals. This corresponds to the map $R \to R\left[t^{\pm 1}\right]$ just being the inclusion, and in particular the grading $R = R_0$. Now consider the ideal

$$R_+ = \bigoplus_{n>0} R_n \ .$$

Then we want to exclude any graded prime ideals $P \supseteq R_+$. Now as a set we can define:

$$\operatorname{Proj} R = \left\{ P \in \operatorname{Spec} R \setminus V\left(R^+\right) \mid P \text{ is graded, } \right\}$$

As a topological space this will just have the subspace topology, but the subtle thing is how this is a scheme.