LECTURE 10
256B

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1. Tensor product of sheaves

Recall we defined the tensor products on sheaves to be the sheafification of the natural presheaf tensor product:

$$\mathcal{M} \otimes \mathcal{N} = \operatorname{sh}\left(\mathcal{M} \otimes_{\mathcal{A}}^{pr} \mathcal{N}\right) \ . \tag{1}$$

This has the universal property:

$$\begin{array}{ccc} \mathcal{M} \times \mathcal{N} & \xrightarrow{\mathcal{A}-\text{bil.}} & \mathcal{L} \\ {\scriptstyle -\otimes-}\downarrow & \overset{\exists !}{\nearrow} & \\ \mathcal{M} \otimes_{\mathcal{A}} \mathcal{N} & & \end{array} \tag{2}$$

just like for ordinary modules. The stalks are just $\mathcal{M}_p \otimes_{\mathcal{A}_p} \mathcal{N}_p$ as we would expect.

1.1. Affine schemes.

Let $X = \operatorname{Spec} R$, $\mathcal{A} = \mathcal{O}_X$, $\mathcal{M} = \tilde{M}$, and $\mathcal{N} = \tilde{N}$. Then we have that

$$\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N} = \left(M \otimes_R N\right)^{\sim} \ . \tag{3}$$

The easiest way to see this is by the universal property of the $\tilde{}$ operation. There is certainly a map

$$M \otimes_R N \to \Gamma\left(X, \tilde{M} \otimes_{\mathcal{O}_X} \tilde{N}\right) \tag{4}$$

which sends $m \otimes n \to m \otimes n$. Then we just want to check this is the identity on stalks. The stalks are

$$\left(M \otimes_R N\right) \otimes_r R_p = M_p \otimes_{R_p} N_p \tag{5}$$

so this is an isomorphism. So this is a good fact to know:

$$\left(M \otimes_R N\right)^{\sim} = \tilde{M} \otimes_{\mathcal{O}_X} \tilde{N} \ . \tag{6}$$

Warning 1. So we have seen that for affines, taking global sections commutes with tensor products. Note that this is not the case for non-affine X. I.e. (for $\mathcal{M}$, $\mathcal{N}$ qco) the following is NOT generally true:

$$\Gamma\left(X, \mathcal{M} \otimes \mathcal{N}\right) = \Gamma\left(X, \mathcal{M}\right) \otimes_{\Gamma(X, \mathcal{O}_X)} \Gamma\left(X, \mathcal{N}\right) \ . \tag{7}$$

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Counterexample 1. There is always a map from the RHS to the LHS above, but it's not always an isomorphism. Let $X = \mathbb{P}^n_k$ and $\mathcal{M} = \mathcal{N} = \mathcal{O}(1)$. Then $\mathcal{M} \otimes \mathcal{N} = \mathcal{O}(2)$. There are of course quasi-coherent, however we can calculate that:

$$\Gamma\left(\mathbb{P}^n_k, \mathcal{O}(d)\right) = k\left[x_0, \cdots, x_n\right]_{(d)} \tag{8}$$

$$\Gamma(\mathcal{M}) = k \cdot \{x_0, \cdots, x_n\} \tag{9}$$

$$\Gamma(\mathcal{M} \otimes \mathcal{N}) = k \cdot \left\{x_0^2, x_0 x_1, \cdots\right\} \ . \tag{10}$$

2. Extension of scalars

Let X and Y be ringed spaces. A map $\varphi \colon X \to Y$ is specified by the data $\left(\varphi, \varphi^{\#}, \varphi^{\flat}\right)$. For $\mathcal{N}$ an $\mathcal{O}_Y$-module, we can take $\varphi^{-1}\mathcal{N}$ which is a $\varphi^{-1}\mathcal{O}_Y$ module. However the map $\varphi^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ doesn't make this an $\mathcal{O}_X$ module unless we tensor it:

$$\varphi^* \mathcal{N} = \mathcal{O}_X \otimes_{\varphi^{-1}\mathcal{O}_Y} \varphi^{-1}\mathcal{N} \ . \tag{11}$$

We should think of this as being like the usual extension of scalars for modules.

Let $\mathcal{M}$ be a sheaf of $\mathcal{O}_X$ modules. Then we want to consider

$$\mathrm{Hom}_{\mathcal{O}_X}\left(\varphi^*\mathcal{N}, \mathcal{M}\right) = \mathrm{Hom}_{\varphi^{-1}\mathcal{O}_Y}\left(\varphi^{-1}\mathcal{N}, \mathcal{M}'\right) \tag{12}$$

where $\mathcal{M}'$ is just $\mathcal{M}$ viewed as a $\varphi^{-1}\mathcal{O}_Y$-module. Since φ^{-1} and φ_* are adjoint, we have

$$\mathrm{Hom}_{\varphi^{-1}\mathcal{O}_Y}\left(\varphi^{-1}\mathcal{N}, \mathcal{M}'\right) = \mathrm{Hom}_{\mathcal{O}_Y}\left(\mathcal{N}, \varphi_*\mathcal{M}\right) \ . \tag{13}$$

This tells us that φ^* is left adjoint to φ_* as functors between $\mathcal{O}_X$-modules and $\mathcal{O}_Y$-modules.[1]

Consider the scheme morphism $\varphi \colon X = \operatorname{Spec} A \to Y = \operatorname{Spec} B$ corresponding to a ring homomorphism $\alpha \colon B \to A$. Then for N a B-module,

$$\varphi^* \tilde{N} = \left(A \otimes_B N\right)^{\sim} \ . \tag{14}$$

A map

$$\left(A \otimes_B N\right)^{\sim} \to \varphi^* \tilde{N} \tag{15}$$

corresponds to a map of A-modules

$$A \otimes_B N \to \left(\varphi^* \tilde{N}\right)(X) \tag{16}$$

which sends $a \otimes n \mapsto a \otimes n$. A stalk of $\left(A \otimes_B N\right)^{\sim}$ looks like $A_P \otimes_{B_Q} N_Q$ where Q is the preimage of P under α. A stalk on the other side is $A_P \otimes_{B_Q} N_Q$ so this is an isomorphism.

There are lots of other ways to see this. One is that we could use the universal property on the right to get a map in the other direction. Another way to do this would be to take a presentation of N

$$B^{(I)} \to B^{(I)} \to N \to 0 \ . \tag{17}$$

This gives a presentation:

$$\mathcal{O}_Y^{(I)} \to \mathcal{O}_Y^{(I)} \to \tilde{N} \to 0 \tag{18}$$

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[1] φ^{-1} was left adjoint to φ_* when dealing with sheaves of sets or abelian groups rather than modules.

and then φ^* gives us

(19)
$$\begin{aligned} \mathcal{O}_X^{(I)} \to \mathcal{O}_Y^{(I)} \longrightarrow \varphi^* \tilde{N} \to 0 \\ A^{(I)} \longrightarrow A^{(I)} \longrightarrow A \otimes_B N \to 0 \end{aligned}$$

and we just compare these. n

2.1. **Generic schemes.** This was just for affine schemes but for any morphisms of schemes $X \to Y$, we can cover Y with affines, take the preimage of these, then cover these preimages, and then just patch everything together. So for all schemes

(20)
$$\varphi^* \left(\mathbf{QCoh}\,(Y) \right) \subseteq \mathbf{QCoh}\,(X) \ .$$

3. Back to Proj

Let $X = \operatorname{Proj} R$, and M be a graded R-module. Then $\tilde{M}$ is an $\mathcal{O}_X$ module. On $X_f = \operatorname{Spec} (R_f)_0$

(21)
$$\tilde{M}\Big|_{X_f} = (M_f)_0^{\sim} \ .$$

Now we can shift degrees

(22)
$$(M\,[d])_n = M_{n+d}$$

to get

(23)
$$\mathcal{O}_X\,(d) = R\,[d]^{\sim} \ .$$

Now suppose that the degree of f divides d, i.e. for some k $\deg f \cdot k = d$. Then

(24)
$$\mathcal{O}_{X_f}\,(d) = (R_f\,[d])_0^{\sim} = (R_f)_d^{\sim}$$

thought of as an $(R_f)_d$-module. But now we actually have an isomorphism of modules:

(25)
$$(R_f)_0 \to (R_f)_d$$

given by multiplication by f^k, and f^{-k} respectively. Therefore we have

(26)
$$\mathcal{O}_{X_f}^{(d)} \cong \mathcal{O}_X|_{X_f} \ .$$

From here on we will assume[2] that the collection of X_f's such that $\deg f = 1$ form a cover, or said differently:

(27)
$$V\,(R_1) = V\,(R_+)$$

(or $V\,(R_1) = \emptyset$ in $X = \operatorname{Proj} R$). This will imply that $\mathcal{O}_X\,(d)$ is:

(1) locally free of rank 1, or
(2) a line bundle[3], or
(3) an invertible sheaf.

[2] This is justified by this thinning argument from a few lectures ago.
[3] This terminology will be justified later.

4. Invertible sheaves

Let X be a ringed space, and let $\mathcal{L}$ be an invertible sheaf, i.e. we can cover X with opens U such that

$$\mathcal{L}|_U \simeq \mathcal{O}_X(U) \ . \tag{28}$$

The nice thing about $\mathcal{O}_X$ is that it has a distinguished section: 1. Under this isomorphism, this will go to $\sigma_U \in \mathcal{L}(U)$. Now we can consider $\mathcal{L}(U \cap V)$. Then we have two sections σ_U and σ_V which are each local generating sections. I.e. there is some g_{UV} such that $\sigma_U = g_{UV}\sigma_V$ and some g_{VU} such that and $\sigma_V = g_{VU}\sigma_U$, i.e. $g_{UV}g_{VU} = 1$, so $g_{UV}g_{VU} \in \mathcal{O}(U \cap V)^\times$ and $g_{VU} = g_{UV}^{-1}$. Then there is some sort of compatibility which says that on $U \cap V \cap W$ we have

$$\sigma_W = g_{VW}\sigma_V = g_{VW}g_{UV}\sigma_U = g_{UW}\sigma_U \tag{29}$$

which means $g_{UW} = g_{VW}g_{UV}$. Given this compatible data, this determines $\mathcal{L}$ completely.

We could also have alternative sections $\sigma'_U = h_U\sigma_U$, where $h_U \in \mathcal{O}(U)^\times$. What is really happening here is something cohomological. The idea is that the σs are the 0-cycles, the choices of these gs are the 1-cycles, and the 2-cycles are the things on triple intersections:

$$Z^1 \to Z^1 \to Z^2 \ . \tag{30}$$

Then the sheaf cohomology

$$H^1\left(X, \mathcal{O}_X^\times\right) = \mathrm{Pic}(X) \ , \tag{31}$$

which is called the Picard group of X, consists of the invertible sheaves on X. This is a group with respect to tensor products.