LECTURE 11
MATH 256B

LECTURE: PROFESSOR MARK HAIMAN
NOTES: JACKSON VAN DYKE

1. Picard group

1.1. Invertible sheaves. Consider some invertible sheaf (line bundle) $\mathcal{L}$ on a ringed space $(X, \mathcal{O}_X)$. Cover X by opens U where we have

$$\mathcal{O}_X|_U \xrightarrow{\simeq} \mathcal{L}|_U \ .$$

In particular, choose an isomorphism $1 \mapsto \sigma_U \in \mathcal{L}(U)$. Then every $\sigma \in \mathcal{L}(U)$ is $f \cdot \sigma_U$ for a unique $f \in \mathcal{O}_X(U)$. So we choose some σ_U for every U in some compatible fashion. On $U \cap V$ we have $\sigma_U, \sigma_V \in \mathcal{L}(U \cap V)$, and for some g_{UV} we have $\sigma_U = g_{UV}\sigma_V$, $\sigma_V = g_{VU}\sigma_U$. This means $\sigma_U = g_{UV} g_{VU} \sigma_U$, so $g_{UV} g_{VU} = 1$. In particular, $g_{UV}, g_{VU} \in \mathcal{O}_X (U \cap V)^\times$ and $g_{VU} = g_{UV}^{-1}$.

On the triple intersections $U \cap V \cap W$ we have

$$\sigma_U = g_{UW}\sigma_W = g_{UV}\sigma_V = g_{UV} g_{VW} \sigma_W$$

which implies

$$g_{UW} = g_{UV} g_{VW} \ .$$

Given all of this data (the g_{UV}s) we can glue copies of $\mathcal{O}_X|_U$ to get $\mathcal{L}$.

Now consider $\mathcal{L} \otimes \mathcal{M}$. Assume we have trivialized them both on the same open covering.[1] So we have $\sigma_U \in \mathcal{L}(U)$ and $\tau_U \in \mathcal{M}(U)$ and

$$\mathcal{O}_X|_U \xrightarrow{\simeq} \mathcal{L}|_U \otimes \mathcal{M}|_U$$

where $1 \mapsto \sigma_U \otimes \tau_U$. Then the structure constants h_{UV} for $\mathcal{L} \otimes \mathcal{M}$ are just $f_{UV} \cdot g_{UV}$ where f_{UV} are the structure constants for $\mathcal{M}$.

If we take $\mathcal{L} = \mathcal{O}_X$, we might as well take $\sigma_U = 1$ and $g_{UV} = 1$. Then for all $\mathcal{M}$, $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X = \mathcal{M}$. We could also define some $\mathcal{L}^\vee$ to have structure constants $h_{UV} = g_{UV}^{-1}$ for the above g_{UV}, and then the product of these will give us that the structure constants are 1. By construction $\mathcal{L}^\vee$ satisfies $\mathcal{L}^\vee \otimes_{\mathcal{O}_X} \mathcal{L} \simeq \mathcal{O}_X$. So we have inverses and a unit with respect to tensor product. In particular, the set of invertible sheaves modulo isomorphism forms the *Picard group* $\mathrm{Pic}(X)$, which is an abelian group with respect to $\otimes$.

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[1]We can do this since given two different coverings we might as well just refine them so they are both trivialized on one refined version.

1.2. **Čech cohomology.** Let $s_U \in \mathcal{O}_X(U)^\times$. Then $1 \mapsto s_U$ defines an isomorphism $\mathcal{O}_X|_U \xrightarrow{\simeq} \mathcal{O}_X|_U$. Then the structure constants will be of the form $g_{UV} = s_U/s_V$. So we can form the following groups:

$$\begin{array}{ccccccc}
C^0 & \longrightarrow & C^1 & \longrightarrow & C^2 & & \cdots \\
\| & & \| & & \| & & \\
\prod_U \mathcal{O}_X(U)^\times & & \prod_{U,V} \mathcal{O}_X(U\cap V)^\times & & \prod_{U,V,W} \mathcal{O}_X(U\cap V\cap W)^\times & & \cdots \\
(s_U) & \longmapsto & g_{UV} = s_U/s_V & \longmapsto & 1 & & \cdots \\
 & & (g_{UV}) & \longmapsto & g_{UW}/(g_{UV}g_{VW}) & & \cdots
\end{array}$$

This is the Čech cohomology of this sheaf with respect to this covering. Then we can define the actual Čech cohomology of the sheaf to be:

$$\lim_{\underline{U}} H^1_{\underline{\underline{U}}}\left(X, \mathcal{O}_X^\times\right) = H^1\left(X, \mathcal{O}_X^\times\right) .$$

The idea is that $H^1\left(X, \mathcal{O}_X^\times\right)$ consists of the invertible sheaves which are trivializable on our covering, where we somehow forget the trivialization. I.e. we identify two trivializations which give the same invertible sheaf up to isomorphism.

1.3. **Digression.** Let X be a $\mathbb{C}$-analytic variety. We can map $e^z : \mathcal{O}_X \to \mathcal{O}_X^\times$. The kernel of this is somehow $\mathbb{Z}$ (or $2\pi\mathbb{Z}$) and this is also surjective:

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X^\times \longrightarrow 0$$

where we regard $\mathbb{Z}$ as the constant sheaf. This gives us a long exact sequence of cohomology:

$$\begin{array}{ccccccc}
H^1(X, \mathcal{O}_X) & \longrightarrow & H^1\left(X, \mathcal{O}_X^\times\right) = \operatorname{Pic}(X) & \longrightarrow & H^2(X, \mathbb{Z}) & \longrightarrow & H^2(X, \mathcal{O}_X) \\
 & & \mathcal{L} & \longmapsto & c(\mathcal{L}) & &
\end{array}$$

where the second map is in fact the Chern class map. For $X = \mathbb{P}^n_{\mathbb{C}}$ for $n \geq 0$ we have $H^1(X, \mathcal{O}_X) = 0$, and $H^2(X, \mathcal{O}_X) = 0$ as well, so in this case we have an isomorphism between the middle terms $H^2(\mathbb{P}^n_{\mathbb{C}}, \mathbb{Z}) = \operatorname{Pic}(\mathbb{P}^n_{\mathbb{C}}) = \mathbb{Z}$. This is the usual complex analytic fact that $\operatorname{Pic}(X) = \mathcal{O}_X(d)$. For k any field[2] the same is true for $\mathbb{P}^n_k$.

1.4. **Back to invertible sheaves.**

Example 1. Let $\mathcal{L} = \mathcal{O}_X(d) = R[d]^\sim$ on $X = \mathbb{P}^n_{\mathbb{A}} = \operatorname{Proj} A[x_0, \cdots, x_n] = \operatorname{Proj} R$. Take the covering $U_i = X_{x_i} = \operatorname{Spec} A\left[\underline{x}, x_i^{-1}\right]_0$ so we have:

$$\mathcal{O}_X(d)|_{U_i} = \left(A\left[\underline{x}, x_i^{-1}\right]_d\right)^\sim .$$

Then we have $x_i^d \in \mathcal{O}_X(d)(U_i)$, and an isomorphism

$$A\left[\underline{x}, x_i^{-1}\right]_0 \xrightarrow{\simeq} A\left[x, x^{-1}\right]_d$$

[2] This isn't true for any commutative ring however.

where $1 \mapsto x_i^d$. In other words, $\sigma_{U_i} = x_i^d$, which means

$$g_{i,j} = \frac{x_i^d}{x_j^d} \ .$$

This makes sense since $\mathcal{O}(U_i \cap U_j) = A\left[\underline{x}, x_i^{-1}, x_j^{-1}\right]_0$.

Notice that if we consider $X = \operatorname{Proj} R/I \subseteq \mathbb{P}_A^n$, the same story holds since we can take the same cover and just intersect them with X.

The special thing about the previous example is that we covered X with X_f such that $\deg(f) = 1$. But this isn't such a restrictive condition as we saw last time. So now we insist that $V(R_1) = V(R_+)$, which is equivalent to $V(R_1) = \emptyset$ in $\operatorname{Proj} R$, which is equivalent to the X_f (for $f \in R_1$) covering X.

Under this assumption, the $\mathcal{O}_X(d)$ are invertible for all d, with $\mathcal{O}_X|_{X_f} \xrightarrow{\simeq} \mathcal{O}_X(d)|_{X_f}$. We can also see right away that

$$\mathcal{O}_X(d) \otimes \mathcal{O}_X(e) = \mathcal{O}_X(d+e) \ .$$

Actually this is a reflection of the following more general fact. Let M and N be two graded R modules, we can compare:

$$\tilde{M} \otimes_{\mathcal{O}_X} \tilde{N} \qquad\qquad (M \otimes_R N)^{\sim} \ .$$

We might expect these to be the same, but let's look a bit closer. On X_f these look like:

$$\left(M_f \otimes_{R_f} N_f\right)_0 \qquad\qquad (M_f)_0 \otimes_{(R_f)_0} (N_f)_0 \ .$$

Of course there is a map in the left direction which sends $m \otimes n \leftarrow\!\shortmid m \otimes n$, but there could be other things on the left. For example we could take $m \in (M_f)_k$ and $n \in (N_f)_{-k}$ and then $m \otimes n$ is on the left. However, if $\deg(f) = 1$ we have that such an $m \otimes n = f^{-k} m \otimes f^k n$. The point is that these actually are isomorphic for $\deg(f) = 1$.

So now if we're in the setting that $V(R_1) = V(R_+)$, then

$$R[f]^{\sim} \otimes_R R[e]^{\sim} = R[d+e]^{\sim} \ .$$

More generally, for any $\mathcal{O}_X$-module $\mathcal{M}$ we can twist it by:

$$\mathcal{M}(d) \coloneqq \mathcal{O}_X(d) \otimes_{\mathcal{O}_X} \mathcal{M} \ .$$

Now we want to look at the global sections:

$$\Gamma_*(\mathcal{M}) = \bigoplus_{d \in \mathbb{Z}} \Gamma(X, \mathcal{M}(d)) \ .$$

This is a graded R-module. The point is that $R_d \to \Gamma(X, \mathcal{O}_X(d))$ and then we can take

$$\Gamma(X, \mathcal{O}_X(d)) \otimes \Gamma(X, \mathcal{M}(e)) \to \Gamma(X, \mathcal{M}(d+e)) \ .$$

So now we have a functor $\tilde{\ }$ from graded R-modules to $\mathcal{O}_X$-modules, and a functor Γ_* from $\mathcal{O}_X$-modules to graded R-modules. Comparing this with the affine world we might expect these to be inverse, but this situation isn't quite so nice. It will turn out to be the case however that if $\mathcal{M}$ is qco, the sheaf associated to $\Gamma_*(\mathcal{M})$ will be $\mathcal{M}$ again as long as the Proj is quasicompact.