LECTURE 12
MATH 256B

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Recall we wanted to set up some situation on $\operatorname{Proj} R$ which is analogous to what we had for $\operatorname{Spec} R$. We had a globalization functor from R-modules to sheaves on $\operatorname{Spec} R$, and a global sections functor from sheaves on $\operatorname{Spec} R$ to R-modules. But even on classical projective space we can't expect the analogous situation to be a one-to-one correspondence since different modules can give the same sheaf. It is however reasonable to believe that under the appropriate assumptions, we could start with a quasi-coherent sheaf, and that there is a way to get a module which will give us the sheaf back.

So let $X = \operatorname{Proj} R$ where R is such that $V(R_1) = \emptyset$. In other words $R_+ \subset \sqrt{R_1}$. If we define $\mathcal{O}_X(d) = R[d]^{\sim}$, then under this assumption these are invertible. We also have

$$\tilde{M} \otimes \mathcal{O}_X(d) = M[d]^{\sim}$$

and more generally

$$\mathcal{O}_X(d) \otimes \mathcal{O}_X(e) = \mathcal{O}_X(d+e) \ .$$

So now let $\mathcal{M}$ be an $\mathcal{O}_X$-module. Then define $\mathcal{M}(d) = \mathcal{O}_X(d) \otimes \mathcal{M}$ and

$$\Gamma_*(\mathcal{M}) := \bigoplus_d \Gamma(X, \mathcal{M}(d)) \ .$$

Note that we could generally define

$$\Gamma(\mathcal{M}, \mathcal{L}) = \bigoplus_{d \in \mathbb{Z}} \Gamma\left(X, \mathcal{M} \otimes \mathcal{L}^{\otimes d}\right)$$

for any line bundle[1] $\mathcal{L}$. Then we claim that $\Gamma_*(\mathcal{M})$ is a graded R-module. We know we have $R_e = R[e]_0 \to \Gamma(\mathcal{O}(e))$, so we have a map

$$R_e \to \Gamma(\mathcal{O}(e)) \otimes \Gamma(\mathcal{M}(d)) \to \Gamma(\mathcal{O}(e) \otimes \mathcal{M}(d)) = \Gamma(\mathcal{M}(d+e))$$

so it is a graded R-module.

We also have that if we start with some other graded R-module M, and take the degree d piece $M_d = M[d]_0$ then we have a morphism

$$M_d \to \Gamma\left(\tilde{M}(d)\right) = \Gamma_*\left(\tilde{M}\right)_d \ .$$

Proposition 1. $\Gamma_* : \mathcal{O}_X\text{-}\mathbf{Mod} \to R\text{-}\mathbf{Mod}_{grd}$ *is right adjoint to* $(\cdot)^{\sim}$.

Proof. This means that morphisms $\tilde{M} \to \mathcal{N}$ are in canonical one-to-one correspondence to morphisms $M \to \Gamma_*(\mathcal{N})$, i.e.

$$\operatorname{Hom}_{\mathcal{O}_X\text{-}\mathbf{Mod}}\left(\tilde{M}, \mathcal{N}\right) = \operatorname{Hom}_{R\text{-}\mathbf{Mod}_{\mathrm{grd}}}(M, \Gamma_*(\mathcal{N})) \ .$$

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[1] For example, an ample line bundle.

Suppose we are given $\tilde{M} \to \mathcal{N}$. Then we apply Γ_*, and use the fact that we have a canonical homomorphism:

$$M \to \Gamma_* \left(\tilde{M} \right) \to \Gamma_* \left(\mathcal{N} \right)$$

so we have a map $M \to \Gamma_* \left(\mathcal{N} \right)$.

Now suppose we are given $\beta : M \to \Gamma_* \left(\mathcal{N} \right)$. To define a sheaf homomorphism, we just need to show what it does on a base of the open sets: the X_fs. On X_f we have $\tilde{M} \Big|_{X_f} = \left(M_f \right)_0^{\sim}$ so we want:

$$\tilde{M} \left(X_f \right) = \left(M_f \right)_0 \to \mathcal{N} \left(X_f \right) \ .$$

We have $\beta (a) \in \Gamma \left(\mathcal{N} (nd) \right) = \Gamma \left(\mathcal{N} \otimes \mathcal{O} (nd) \right)$, and $f \in \Gamma \left(\mathcal{O} (d) \right)$ is a generating section on X_f, so it is invertible. We want to think of this as landing in

$$f^{-n} \to \Gamma \left(X_f, \mathcal{O} (-nd) \right) \ .$$

Now we can tensor these to get a section

$$\beta (a) f^{-n} \in \Gamma \left(X_f, \mathcal{N} \right)$$

as desired. □

We really want to know that qco sheaves on a Proj correspond to modules. The issue is that this isn't true unless we make an additional assumption that $X = \operatorname{Proj} R$ is quasicompact.[2] In particular, we can cover it by finitely many X_fs.

Theorem 1. *Under the above assumptions, if $\mathcal{M}$ is a qco $\mathcal{O}_X$-module, then this implies that $M = \Gamma_* \left(\mathcal{M} \right)^{\sim}$.*

Proof. We are in the following situation:

$$\begin{array}{ccccc} X & \xleftarrow{\quad \pi \quad} & U & \xhookrightarrow{\quad j \quad} & Y = \operatorname{Spec} R \\ \| & & \| & & \\ \operatorname{Proj} R & & Y \setminus V \left(R_+ \right) & & \end{array} .$$

$$X_f = \operatorname{Spec} \left(R_f \right)_0 \longleftarrow Y_f = \operatorname{Spec} R_f$$

Recall we should think of U as a principle $\mathbb{G}_m$ bundle, and we should think of X as the quotient space:

$$\begin{array}{ccc} Y_f & = & \mathbb{G}_m \times X_f = \operatorname{Spec} S \left[x^{\pm 1} \right] \\ \downarrow & & \\ X_f & = & \operatorname{Spec} S \end{array}$$

where $S \left[x^{\pm 1} \right] \cong R_f$ under $x \mapsto f$. Then the idea is that we want

$$\Gamma_* \left(\mathcal{M} \right) = \Gamma_Y \left(j_* \pi^* \mathcal{M} \right) \ .$$

We know that

$$\pi_* \mathcal{O}_Y = \bigoplus_{d \in \mathbb{Z}} \mathcal{O}_X (d) \ .$$

[2] Note we still maintain that $V \left(R_1 \right) = \emptyset$.

Now we can calculate:

$$\begin{aligned}\Gamma_*(\mathcal{M}) = \bigoplus_{d\in\mathbb{Z}} \Gamma(X, \mathcal{M}\otimes\mathcal{O}(d)) &= \Gamma\left(X, \bigoplus_{d\in\mathbb{Z}} \mathcal{M}\otimes\mathcal{O}(d)\right) \\ &= \Gamma(X, \mathcal{M}\otimes\pi_*\mathcal{O}_Y) \\ &= \Gamma(X, \pi_*\pi^*\mathcal{M}) \\ &= \Gamma_U(\pi^*\mathcal{M}) \\ &= \Gamma_Y(j_*\pi^*\mathcal{M}) \ .\end{aligned}$$

The issue is that there is a problem with this calculation, because global sections do not necessarily commute with direct sums. They do however commute with products, so we generally have a containment:

$$\begin{array}{ccc} \Gamma\left(\prod_\alpha M_\alpha\right) & = & \prod_\alpha \Gamma(M_\alpha) \\ \uparrow & & \uparrow \\ \Gamma\left(\bigoplus_\alpha M_\alpha\right) & \hookleftarrow & \bigoplus_\alpha \Gamma(M\alpha) \end{array} \ .$$

However, Γ does preserve direct sums for qco sheaves on affine schemes. But now we could have a global section for which on each piece of the open cover only finitely many are nonzero but globally infinitely many are nonzero. This is not a problem however if the affine open cover is finite. I.e. if the M_α are qco, and X is quasicompact, then Γ does commute with direct sums. Therefore the above calculation was correct, and now we just have to use the fact that j is a quasicompact morphism which preserves qco sheaves to finish up.

To be continued...

□