LECTURE 13
MATH 256B

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We will continue with the proof we were doing last time. Recall the theorem was the following:

Theorem 1. *Let* $X = \operatorname{Proj} R$ *be quasicompact and* $V(R_1) = \emptyset$. *If* $\mathcal{M} \in \mathbf{QCoh}(X)$, *then* $\Gamma_*(\mathcal{M})^{\sim} = \mathcal{M}$.

Continued proof. Recall that we had $X \leftarrow U$ for some open set $U \subseteq Y = \operatorname{Spec} R$ where $U = Y \setminus V(R_+) = Y \setminus V(R_1)$, so we can cover U with the preimages of the X_fs (for $\deg f = 1$) so this map $\pi : U \to X$ is a principal $\mathbb{G}_m$ bundle as we have seen before. Locally we have

$$X_f = \operatorname{Spec} S = \operatorname{Spec}(R_f)_0 \leftarrow U_f = \operatorname{Spec} R_f = \operatorname{Spec} S\left[f^{\pm 1}\right]$$

and geometrically $U_f = \mathbb{G}_m \times X_f$. The conclusion from this picture is that

$$\pi_* \mathcal{O}_U = \bigoplus_{d \in \mathbb{Z}} \mathcal{O}_X(d) \ .$$

The point here is the following. Rank n vector bundles are bundles with fibers consisting of n-dimensional vector spaces. Then we can pass to the associated GL_n bundle where the fibers are now the groups GL_n of invertible transformations on the fibers of the vector bundle. So somehow these notions are equivalent, and in this case we're seeing that line-bundles are somehow the same as $\mathrm{GL}_1 = k^\times = \mathbb{G}_m$-bundles.

So the above only used that $V(R_1) = \emptyset$, and then the other thing we observed last time used the quasi-compactness. In particular we saw that since X is quasicompact and $\mathcal{M}$ is qco, we have

$$\Gamma_*(\mathcal{M}) = \bigoplus_d \Gamma(X, \mathcal{M} \otimes \mathcal{O}(d)) = \Gamma\left(X, \bigoplus_d \mathcal{M} \otimes \mathcal{O}(d)\right) \ .$$

Then since $\oplus_d M \otimes \mathcal{O}(d) = \mathcal{M} \otimes \pi_* \mathcal{O}_U$ we have that

$$\Gamma_*(\mathcal{M}) = \Gamma(X, \mathcal{M} \otimes \pi_* \mathcal{O}_U) = \Gamma(U, \pi^* \mathcal{M}) \ .$$

By the definition of the direct image functor this is really:

$$\Gamma(U, \pi^* \mathcal{M}) = \Gamma(Y, j_* \pi^* \mathcal{M}) \ .$$

Now we notice that π^* preserves qco sheaves, and since j is quasicompact and separated, we have that $j_* \pi^* \mathcal{M}$ is quasi-coherent as well. So we have a qco sheaf on the affine scheme Y, which has global sections equal to $\Gamma_*(\mathcal{M})$. I.e. $j_* \pi^* \mathcal{M} = \Gamma_*(\mathcal{M})^{\sim_{\text{aff}}}$ as a sheaf on Y. Then we have

$$\pi^* \mathcal{M} = \Gamma_*(\mathcal{M})^{\sim_{\text{aff}}}|_U$$

Date: February 22, 2019.

on $U_f = \operatorname{Spec} R_f$, which means

$$\pi^* \mathcal{M}|_{U_f} \leftrightarrow M_f^{\sim}$$

where we write $M = \Gamma_* (\mathcal{M})$. But it's a general fact that

$$\pi_* \pi^* \mathcal{M} = \mathcal{M} \otimes \pi_* \mathcal{O}_U$$

which means

$$\pi_* \pi^* \mathcal{M}|_{X_f} = M_f^{\sim}$$

where we think of M_f as a module for $(R_f)_0$. But we also know that

$$M_f^{\sim} = \mathcal{M} \oplus \bigoplus_d \mathcal{O}(d)$$

and in particular,

$$\mathcal{M}|_{X_f} = (M_f)_0^{\sim} \ .$$

Now we would need to check that these are compatible on all the X_fs, but this is sort of trivial from the canonical way we did this. □

Under the above assumptions, we now have the following situation. We have a functor $\tilde{\cdot}$ with left inverse Γ_* (which is also right adjoint):

$$R\text{-}\mathbf{Mod}_{\text{grd}} \underset{\Gamma_*}{\overset{\widetilde{(\cdot)}}{\rightleftarrows}} \mathbf{QCoh}(X) \ .$$

In the affine case, this was an equivalence, but in this case Γ_* is not, and in fact it is not even exact.

Consider any closed $i : Z \hookrightarrow X = \operatorname{Proj} R$. Then we get a surjective map $\mathcal{O}_X \twoheadrightarrow i_* \mathcal{O}_Z$ with kernel $\mathcal{I}(Z)$. Note these are all quasi-coherent sheaves. So we have a SES

$$0 \to \mathcal{I}(Z) \to \mathcal{O}_X \to i_* \mathcal{O}_Z \to 0 \ ,$$

which we can apply Γ_* to. This is always left exact, so we get:

$$0 \to \Gamma_* (\mathcal{I}(Z)) \to \Gamma_* (\mathcal{O}_X) \to \Gamma_* (\mathcal{O}_Z)$$

where we have identified $\mathcal{O}_Z$ with $i_* \mathcal{O}_Z$. In particular we have the following:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Gamma_* (\mathcal{I}(Z)) & \longrightarrow & \Gamma_* (\mathcal{O}_X) & \longrightarrow & \Gamma_* (\mathcal{O}_Z) \\ & & & & \uparrow & \nearrow & \\ 0 & \longrightarrow & J & \hookrightarrow & R & & \end{array}$$

but the dashed map could easily not be surjective.

Exercise 1. Show that J is such that $V(J) = Z$.

Example 1. Consider $X = \mathbb{P}^n_R$ for $R = k[x_0, \cdots, x_n]$ for $n > 0$. Then take $Z \hookrightarrow \mathbb{P}^n_R$ so there is some $J \subset k[\underline{x}]$ such that $Z = \operatorname{Proj} k[\underline{x}] / J$. Then we have

$$0 \to J \to R \to \Gamma_* (\mathcal{O}_Z)$$

but this last map isn't typically surjective. The simplest example is to just take Z to be a point. In this example we have

$$\Gamma_* (\mathcal{O}_Z) = \bigoplus_{d \in \mathbb{Z}} k$$

so it has nonzero stuff in negative degrees, so the map certainly can't be surjective in negative degrees.

This raises the following question. We have seen that localization is somehow surjective under the above assumptions, but not injective, since there will be different graded R-modules which give rise to the same qco sheaf. This is somehow saying there is a kernel here. So suppose we have a graded R-module homomorphism $M \to N$. Then we have an induced map of quasi-coherent sheaves on X $\tilde{M} \to \tilde{N}$. Now if Γ_* was actually inverse to localization then the ring homomorphism being an isomorphism would imply the sheaf map is an isomorphism. But the second could be an isomorphism without the first one being an isomorphism. So let's consider this situation. I.e. we have an exact sequence

$$0 \to K \to M \to N \to Q \to 0$$

where K and Q are nonzero, however after applying localization we have the following:

$$0 \to \tilde{K} = 0 \to \tilde{M} \xrightarrow{\sim} \tilde{N} \to \tilde{Q} = 0 \to 0 \ .$$

Indeed, if Γ_* were an equivalence we would see that nonzero modules must have a nonzero associated sheaf, but this is not the case. The idea here is that we can detect the sense in different modules can give rise to the same sheaf by understanding the sense in which nonzero modules can give the zero sheaf. So we want to understand the question:

Question 1. Which modules M have $\tilde{M} = 0$?

Cover X by

$$\bigcup_{i=1}^{n} X_{f_i} = X$$

for $f_i \in R_1$. Then $(M_{f_i})_0^{\sim} = 0$ implies $(M_{f_i})_0 = 0$, which implies $(M_{f_i})_d \cong M_{f_i}$ are all 0, so for all i, we have $M_{f_i} = 0$.

But this says that for all $a \in M$ some power of each of the f_is will kill a, i.e. some power of the ideal $(f_1, \cdots, f_n)$ kills a. But now if we additionally suppose R_+ itself is finitely generated, then we could have taken the f_i to consist of degree 1 generators of R_+, so WLOG $(f_1, \cdots, f_n) = R_+$, so every element of M is killed by some power of R_+. So now suppose further that M is finitely generated. This means that $\tilde{M} = 0$ iff some power of R_+ kills M, which is equivalent to saying that $M_d = 0$ for $d \gg 0$.

So we always have a map $M \to \Gamma_*\left(\tilde{M}\right)$, but now

$$\tilde{M} \xrightarrow{\simeq} \Gamma_*\left(\tilde{M}\right)^{\sim}$$

so under appropriate finiteness assumptions (M f.g. and $\Gamma\left(\tilde{M}\right)$ f.g.[1]) then $M_d \xrightarrow{\simeq} \Gamma_*\left(\tilde{M}\right)_d$ for $d \gg 0$.

This tells us that morally (modulo some technical things[2]) we should think of Γ_* as an inverse of localization in high degrees, and in particular exact in high degrees.

[1]It is not obvious, but M being f.g. implies $\Gamma\left(\tilde{M}\right)$ is f.g. by the assumptions in the theorem, i.e. qco and quasi-compact.

[2]Probably something like coherent sheaves on Proj of a Noetherian ring or something like this.

This is related to Serre's vanishing theorem which says that if $\mathcal{M}$ is coherent on nice $X = \operatorname{Proj} R$ then this implies the higher sheaf cohomology is

$$H^i\left(X, \mathcal{M} \otimes \mathcal{O}_X(d)\right) = 0$$

for $i > 0$ and $d \gg 0$. This is a fundamental fact about cohomology of sheaves on projective space.