LECTURE 14
MATH 256B

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1. Representing Proj R as a functor

Previously we talked about the idea that as long as $V(R_1) = \emptyset$, then the map $U = \operatorname{Spec} R \backslash V(R_+) \to X = \operatorname{Proj} R$ looks like a principle $\mathbb{G}_m$ bundle. In particular, $\mathbb{G}_m$ invariant morphisms $U \to Y$ are the same as morphisms $X \to Y$:

$$\begin{tikzcd} X \arrow[r] & Y \\ U \arrow[ur, "\mathbb{G}_m \text{ invariant}"'] & \end{tikzcd} .$$

This entirely characterizes the Proj, but it somehow only tells us the functor it co-represents, so it doesn't really help us understand morphisms $T \to X$.

Also recall that under the assumption $V(R_1) = \emptyset$, the $\mathcal{O}_X(d)$ are invertible, and they comprise a subgroup of the Picard group under the operation

$$\mathcal{O}_X(d) \otimes \mathcal{O}_X(e) = \mathcal{O}_x(d+e)$$

which is generated by $\mathcal{O}_X(1)$.

Given a morphism $\varphi : T \to X$ we get:

(i) The invertible sheaf $\mathcal{L} = \varphi^* \mathcal{O}(1)$ on T.[1] Note also that
$$\mathcal{L}^{\otimes d} = \varphi^* \mathcal{O}(d) \ .$$

(ii) $f \in R_d$ gives a global section in $\Gamma(X, \mathcal{O}_X(d))$ and then we get a global section in $\Gamma\left(T, \mathcal{L}^{\otimes d}\right)$. So if we define
$$\Gamma_+(T, \mathcal{L}) = \bigoplus_{d \geq 0} \Gamma\left(T, \mathcal{L}^{\otimes d}\right) \ ,$$
we get a graded ring homomorphism
$$\alpha : R \to \Gamma_+(T, \mathcal{L}) \ .$$

(iii) These data are such that $\mathcal{L}$ is generated by its (global) sections which are in $\alpha(R_1)$. This means the following. These global sections can be thought of as homomorphisms $\mathcal{O} \to \mathcal{L}$ which send $1 \mapsto s$. On an open subset this means $\mathcal{O}|_U \xrightarrow{\simeq} \mathcal{L}|_U$ given by $1 \mapsto s$.

Theorem 1. *All data*[2] *(i) and (ii) satisfying (iii) come from a unique morphism* $\varphi : T \to X$.

Date: February 25, 2019.

[1] More generally the pullback of a vector bundle is a vector bundle.

[2] $\mathcal{L}$ and $R \to \Gamma_+(\mathcal{L})$ up to isomorphism.

Remark 1. We can sort of rigidify this in the following sense. Part of this data is the map $R_0 \to \Gamma\left(T, \mathcal{L}^{\otimes 0}\right) = \Gamma(T, \mathcal{O})$, so we can tensor $R \otimes_{R_0} \mathcal{O}(T)$ which maps to

$$R \otimes_{R_0} \mathcal{O}(T) \to \bigoplus_{d \geq 0} \mathcal{L}^{\otimes d} \ .$$

In these terms the condition (iii) says that this map is surjective in degree 1, so giving this up to isomorphism is giving the kernel of this. If we didn't want to think of this as data up to isomorphism we could think of this as a theorem about ideals in this graded algebra such that the quotient is isomorphic to this direct sum of tensor powers of the degree 1 part (which is an invertible sheaf).

2. Examples

2.1. Projective space.

2.1. Projective space. Let $k = \overline{k}$ and consider $\mathbb{P}^n_k$. This is of course a scheme over k, so anything that maps to it is a scheme over k. We want to see how the abstract discussion above becomes concrete in this example. Consider $T \to \mathbb{P}^n_k = \operatorname{Proj} k[x_0, \cdots, x_n]$. Consider $\mathcal{L}$ an invertible sheaf on T.

Now let's make sense of what it means for a section to generate $\mathcal{L}$ on an open set. So let $s \in \Gamma(T, \mathcal{L})$. Just as if s was an honest function, there is a closed subscheme $V(s)$, then we can consider the complement $W \subseteq T$. Locally there will be various opens U on which $\mathcal{L}|_U \simeq \mathcal{O}|_U$ where $s \mapsto f$. But the f depends on the choice of isomorphism. I.e. we could make a different choice to get some f'. These differ by an isomorphism $\mathcal{O}|_U \simeq \mathcal{O}|_U$ given by some $1 \mapsto h$, which has inverse map given by $h' \mapsfrom 1$, but we must have $h' = h^{-1}$ which means $h \in \mathcal{O}^\times(U)$. In particular, since $f' = hf$, we have $V(f) = V(f')$. So we have a well-defined closed subscheme.

More abstractly (but maybe cleaner) we can think of s as a map $s : \mathcal{O} \to \mathcal{L}$ which sends $1 \mapsto s$. Then we can tensor this with its dual $\mathcal{L}^\vee \otimes -$ which yields

$$\mathcal{L}^\vee \xrightarrow{s'} \mathcal{O} \ .$$

The image $I \subseteq \mathcal{O}$ is a locally principal qco ideal sheaf. So now we can talk about $V(I)$, which is just $V(s)$. So we end up with the same picture.

There is even a third way to look at this. We can look at this map, and continue it to its cokernel $\mathcal{A}$:

$$\mathcal{O} \to \mathcal{L} \to \mathcal{A} \to 0$$

and we can tensor it again with $\mathcal{L}^\vee$ to get

$$\mathcal{L} \to \mathcal{O} \to \mathcal{O}/I \to 0$$

since $\mathcal{O}/I = \mathcal{L}^\vee \otimes \mathcal{A}$. Now we can look at the support of $\mathcal{A}$. In general the support of a sheaf doesn't have to be closed but the support of a section of a sheaf is closed. Since the support of the whole sheaf is the union of the supports of its sections, and its enough to take generating sections we have that $\operatorname{Supp}(\mathcal{A})$ is closed. In addition, tensoring with $\mathcal{L}^\vee$ doesn't change this, so

$$\operatorname{Supp}(\mathcal{A}) = \operatorname{Supp}(\mathcal{O}/I) = V(I) \ .$$

From this we can see that its complement is the locus of points where s generates $\mathcal{O}$ in a stalk. So let $W = T \setminus V(s)$. Then this map $s : \mathcal{O} \to \mathcal{L}$ is surjective on W, but since this is an invertible sheaf, it is actually an isomorphism, and this is clearly the largest open subset on which it could be. So s generates $\mathcal{L}$ on W in two senses. It's an isomorphism, and in every point of W the germ of s is a generator of $\mathcal{L}$.

Now let's get back to projective space. So we have this ring homomorphism $\alpha : k[x_0, \cdots, x_n] \to \Gamma_+(\mathcal{L})$. We know what happens in degree 0, so we just have to specify the image of the x_i. Since they have no relations in the polynomial ring, we can send them anywhere arbitrarily:

$$\alpha(x_i) = s_i \in \Gamma(T, \mathcal{L}) \ .$$

Then $U_i = T \setminus V(s_i)$ over T.

On U_i, $s_j = f_{ji} s_i$ for some f_{ji} on U_i. Then we get a sort of ratio of these

$$(s_0 : s_1 : \cdots : s_n) = (f_{0i} : f_{1i} : \cdots : 1 = f_{ii} : \cdots : f_{ni}) \ .$$

The point is that sections of an invertible sheaf are functions, but ratios of them are also somehow functions as long as the one in the denominator is a generator. In particular we can evaluate at any point, to get a map

$$p \in T \mapsto (s_0 : \cdots : s_n)(p) \ .$$

Geometrically, at a point p, the fiber of the sheaf of functions is values of functions at p, which is a line, and the fiber of $\mathcal{L}$ is also a line. But for functions it has an origin and a scale (a value 1), and $\mathcal{L}$ is a line with an origin but no unit. So the point is that if we have two sections of $\mathcal{L}$, their values in the fiber only really have a well-defined ratio.

Example 1. Consider $\mathbb{P}^k \subseteq \mathbb{P}^n$ given by $V(x_{k+1}, \cdots, x_n)$. We could also consider $V(x_0, \cdots, x_k)$ which looks like $\mathbb{P}^{n-1-k}$. Notice that these are disjoint. Now suppose we take some $\mathbb{P}^{k+1}$ containing our $\mathbb{P}^k$. Then there is exactly one point where this meets $\mathbb{P}^{n-1-k}$. In particular we can take some point in $\mathbb{P}^n \setminus \mathbb{P}^k$. Then $\mathbb{P}^k$ and this point are on some $\mathbb{P}^{k+1}$, which meets $\mathbb{P}^{n-1-k}$ in a single point. So we have a map $\mathbb{P}^n \setminus \mathbb{P}^k \to \mathbb{P}^{n-1-k}$, but is this a morphism?

Well if it's supposed to be a morphism to a projective space, it should be specified by a line bundle and an appropriate number of sections with no common zero locus. Take $\mathcal{L} = \mathcal{O}(1)$ for the line bundle. Then we need $n-k$ sections $s_{k+1}, \cdots, s_n$ of this which are generating sections on T. The point is that these are just $s_i = x_i$. These are sections with zero loci $V(x_i)$, and the complement of this $\mathbb{P}_k$ is the union of the open sets where these are generators. I.e. these generate $\mathcal{O}(1)$ on T.