LECTURE 15
MATH 256B

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Recall we are considering $X = \operatorname{Proj} R$ where $V(R_1) = \emptyset$ and we want to understand the functor $\underline{\underline{X}}$. In particular, given $\varphi : T \to X$ and $\mathcal{L} = \varphi^* \mathcal{O}(1)$, we claim that

$$\underline{\underline{X}}(T) = \{\mathcal{L}, R \to \Gamma_+(\mathcal{L})\} / \cong$$

where

$$\Gamma_+(\mathcal{L}) \coloneqq \bigoplus_{d \geq 0} \Gamma\left(T, \mathcal{L}^{\otimes d}\right) .$$

Last time we saw how such a φ can give us $\mathcal{L}$ and α, and today we want to go in the opposite direction. But first we will see some examples.

1. Examples

Example 1. Recall last time we were considering $\mathbb{P}^n_K$ for a field K. Then take $\mathbb{P}^k = V(x_{k+1}, \cdots, x_n)$ living inside this, and write $U = \mathbb{P}^n \setminus \mathbb{P}^k$. The map $U \to \mathbb{P}^{n-1-k}$ is given by the following. We can take some point in U, and then $\mathbb{P}^k$ and this point are on some $\mathbb{P}^{k+1}$, which meets $\mathbb{P}^{n-1-k}$ in a single point. So we have a map $U \to \mathbb{P}^{n-1-k}$. Then for this to be a morphism we need a line bundle and some sections. The line bundle is $\mathcal{L} = \mathcal{O}(1)|_U$, and the sections $s_{k+1}, \cdots, s_n \in \mathcal{L}(U)$ are just $x_{k+1}, \cdots, x_n$. U is the biggest open set on which these comprise a generating collection.

Example 2. Let C be a projective curve (closed in $\mathbb{P}^n$, irreducible, smooth). Any hyperplane, i.e. a $\mathbb{P}^{n-1}$, will meet C in some points. Then we can find some $\mathbb{P}^{n-2}$ inside this which avoids C. Then for any point in C, the $\mathbb{P}^{n-1}$ which contains both this point and our $\mathbb{P}^{n-2}$ will intersect a $\mathbb{P}^1$ at exactly 1 point.[1] So we have a projection map from C to $\mathbb{P}^1$.

As long as we make generic choices (i.e. we don't set it up exactly wrong) the map from C to $\mathbb{P}^1$ will be sort of finite to one, i.e. the inverse image of any point on $\mathbb{P}^1$ is some finite set of points. It will also turn out to be surjective. This gives us a ramified covering of the projective line by any curve. This is a way of describing curves basically up to isomorphism.

Example 3. Let C be $V\left(xz - y^2\right)$. This is the projective closure of $x = y^2$. So choose the $\mathbb{P}^{n-1} = \mathbb{P}^1$ to be the y-axis, and then we take the $\mathbb{P}^{n-2} = \mathbb{P}^0$ avoiding the curve to be $(0 : 1 : 0) = (x : y : z)$. Then we connect points on C to this point and these intersect the x axis. So this is how we project. So we get a ramification point at the origin, and at the point at infinity. So the conic C maps to $\mathbb{P}^1$ which is a double cover ramified at two points.

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[1] The idea is that $\mathbb{P}^{n-1}$ looks like $V(x_n)$ and $\mathbb{P}^1$ looks like $V(x_0, \cdots, x_{n-1})$.

The Hurwitz number $h_{g,k_1,\dots,k_n}$ is the count of ramified coverings of $\mathbb{P}^1$ which are connected curves of genus g with n numbered preimages of the point at infinity having multiplicities $k_1, \cdots, k_n$. If we know the monodromy we can compute these. As it turns out $h_{g,k_1,\dots,k_n}$ is an intersection number on $\overline{m_{g,n}}$, which is somehow a moduli space of stable curve of genus g with n marked points.

Example 4. Let $R = A[x]$. Then $X = \operatorname{Proj}(R) = X_x$ is affine. In particular it is $\operatorname{Spec} A\left[x^{\pm 1}\right]_0 = \operatorname{Spec}(A)$. Therefore we already know what its functor is:

$$\underline{\underline{X}}(T) = \{A \to \mathcal{O}_T(T)\} \ . \tag{1}$$

So let's see if this more complicated description above agrees with this. We should have a line bundle $\mathcal{L}$ on T, and a morphism $A[x] \xrightarrow{\alpha} \Gamma(\mathcal{L})$. As a part of this there is

$$\alpha_0 : A \to \Gamma(T, \mathcal{O}_T)$$

which is the morphism in (1). The image of x:

$$\alpha(x) = s \in \Gamma(T, \mathcal{L})$$

is a section of $\mathcal{L}$, and in particular it is a generating section. But if a line bundle has a global section which generates it, then we have an isomorphism:

$$\begin{array}{c} \mathcal{O} \xrightarrow{\simeq} \mathcal{L} \\ 1 \longmapsto s \end{array} \ .$$

So these points of view do agree in this case.

2. How to prove this

So we want to see how given $\mathcal{L}$ and $\alpha : R \to \Gamma_+(\mathcal{L})$ (such that $\alpha(R_1)$ generates $\mathcal{L}$) we can get $\varphi : T \to X$. There is a canonical morphism $\psi : T \to \operatorname{Proj}\Gamma_+(\mathcal{L})$ (assuming $\mathcal{L}$ is globally generated). For $s \in \Gamma(T, \mathcal{L})$ we get $T_s = T \setminus V(s)$ and s generates $\mathcal{L}|_{T_s}$. In particular we have an isomorphism

$$\mathcal{O}|_{T_s} \simeq \mathcal{L}|_{T_s}$$

which sends $1 \mapsto s$. Then we can tensor with $\mathcal{L}^\vee$ to get the inverse of this map:

$$s \otimes \mathcal{L}^\vee : \mathcal{L}^\vee \xrightarrow{\simeq} \mathcal{O} \ .$$

We write $\mathcal{L}^{-\otimes n} = \mathcal{L}^{\vee \otimes n}$. So we have $s^{-1} \in \mathcal{L}^\vee(T_s)$ and $s \in \Gamma_+(\mathcal{L})_1$. Then g/s^d maps into $Y_s = \operatorname{Spec}(\Gamma_+(\mathcal{L})_s)_0$ where $g \in \Gamma\left(T, \mathcal{L}^{\otimes d}\right)$ and $s^{-d} \in \Gamma\left(T_s, \mathcal{L}^{-\otimes d}\right)$. So we can view $g/s^d \in \mathcal{O}_T(T_s)$. Now we have a map

$$\left(\Gamma_+(\mathcal{L})_s\right)_0 \to \mathcal{O}_T(T_s)$$

which means we have a map

$$T_s \to \operatorname{Spec}\left(\Gamma_+(\mathcal{L})_s\right)_0 = Y_s$$

so these fit together and we get a morphism $T \to Y$. In particular this factors through $W = \bigcup_s Y_s$:

$$\begin{array}{ccc} T & \longrightarrow & Y \\ & \searrow & \uparrow \\ & & W \end{array} \ .$$

The point is that the union of the Y_ss might not cover Y, but the union of the T_ss do cover T, so we get this map.

Now we're almost done. For $R \to \Gamma_+(\mathcal{L})$ we get a map between opens

$$U \to X = \operatorname{Proj} R$$

where $Y \supset U$ and $U = Y \setminus V(\alpha(R_+)) = Y \setminus V(\alpha(R_1))$. But since $\alpha(R_1)$ generates $\mathcal{L}$, the map $T \to Y$ actually maps $T \to U$:

$$\begin{array}{ccc} & & U \\ & \nearrow & \uparrow \\ T & \longrightarrow & Y \end{array} .$$

Now we have to check a bunch of things, but it all works out. So we get a map $\varphi : T \to X$ as desired.