LECTURE 16

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Today we will do some classical algebraic geometry. We will consider two projective varieties, and we want to see if their product is a projective variety. In particular we want to embed this in projective space. This construction is called the Segré embedding.

1. Segré embedding

1.1. **Classical picture.** Consider two projective varieties $X \subseteq \mathbb{P}_k^l$ and $Y \subseteq \mathbb{P}_k^m$. Then the embedding is given by:

$$X \times_k Y \subseteq \mathbb{P}_k^l \times \mathbb{P}_k^m \xhookrightarrow{S} \mathbb{P}^n = \mathbb{P}^{(l+1)(m+1)-1}$$

$$(x_0 : \cdots : x_l), (y_0 : \cdots : y_m) \longmapsto (z_{00}, \cdots, z_{lm})$$

where

$$z_{ij} = x_i y_j \ .$$

This is a large space we're embedding this in, but if we take any fewer dimensions we could hit the point $(0 : \cdots : 0)$.

Supposing that this is a closed embedding, this should correspond to some homogeneous ideal I in the z_{ij}s such that the image of S is $V(I)$. Some things that are certainly in I look like

$$I \ni z_{ij} z_{i'j'} - z_{i'j} z_{ij'} = x_i x_{i'} y_i y_{i'} \ .$$

Another way of describing these elements is as the 2×2 minors of the following matrix:

$$\begin{pmatrix} y_0 \\ \cdots \\ y_m \end{pmatrix} \begin{pmatrix} x_0 & \cdots & x_l \end{pmatrix} = \begin{pmatrix} z_{00} & \cdots & z_{l0} \\ \vdots & \ddots & \vdots \\ z_{0m} & \cdots & z_{lm} \end{pmatrix} \ .$$

But this might not be the whole ideal.

We can view this situation as follows.

$$\begin{aligned} \mathbb{P}_k^l &= \operatorname{Proj} k[x_0, \cdots, x_l] = \operatorname{Proj} R \\ \mathbb{P}_k^m &= \operatorname{Proj} k[y_0, \cdots, y_l] = \operatorname{Proj} S \\ \mathbb{P}_k^n &= \operatorname{Proj} k[z_{00}, \cdots, z_{lm}] \ . \end{aligned}$$

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Now we can map:

$$\begin{aligned} k[\underline{z}] &\longrightarrow \left(k[\underline{x}] \otimes_k k[\underline{y}]\right)_\Delta \\ z_{ij} &\longmapsto x_i y_j \end{aligned} \tag{1}$$

where

$$R \otimes_k S = \bigoplus_{(d,e)\in\mathbb{N}^2} R_d \otimes S_e$$

is $\mathbb{N} \times \mathbb{N}$ graded with diagonal

$$(R \otimes_k S)_\Delta = \bigoplus_{d\in\mathbb{N}} R_d \otimes S_d \ .$$

As it turns out the map in (1) is surjective. This is an example of the following theorem which will tell us that that Proj of this really is the product of the corresponding schemes.

2. Main theorem

Theorem 1. *Let R and S be graded A-algebras where A has degree 0. So $X = \operatorname{Proj} R$ and $Y = \operatorname{Proj} S$ are graded schemes over A. Then*

$$X \times_A Y \cong \operatorname{Proj}\left((R \otimes_A S)_\Delta\right) \ .$$

Remark 1. If $V(R_1), V(S_1) = \emptyset$ we have

$$\mathcal{L} = \mathcal{O}_X(1) \otimes \mathcal{O}_Y(1) \ .$$

and a map

$$R_d \otimes S_d \to \Gamma\left(\mathcal{O}_X(d) \otimes \mathcal{O}_Y(d)\right) = \Gamma\left(\mathcal{L}^{\otimes d}\right)$$

so we get a map

$$(R \otimes_A S)_\Delta = \Gamma_+(\mathcal{L}) \ .$$

Now if we wanted to continue proving it in this case we would have to construct an inverse map, i.e. we need the line bundles:

$$\mathcal{M} = \mathcal{O}_X(1) \otimes \mathcal{O}_Y(0) \qquad \mathcal{N} = \mathcal{O}_X(0) \otimes \mathcal{O}_Y(1) \ .$$

To get these we form the $R \otimes_A S$-module

$$M = (R[d] \otimes_A S[e])_\Delta \ ,$$

and then we can check that

$$M^\vee = \mathcal{O}_X(d) \otimes \mathcal{O}_Y(e) \ .$$

The theorem holds for the general case though, so we will prove that version in more detail.

Proof. Let $Z = \operatorname{Proj}\left((R \otimes_A S)_\Delta\right)$. We know X is covered by the X_f for $f \in R_d$, and Y is covered by the Y_g for $g \in R_e$ where

$$X_f = \operatorname{Spec}(R_f)_0 \qquad Y_g = \operatorname{Spec}(S_g)_0 \ .$$

So we have

$$X_f \times_A Y_g = \operatorname{Spec}\left((R_f)_0 \otimes_A (S_g)_0\right) \ .$$

Now define[1]

$$h = f^e \otimes g^d \in (R \otimes_A S)_\Delta \ .$$

Then

$$Z_h = \operatorname{Spec}\left(((R \otimes_A S)_d)_{(0,0)} \right) \ ,$$

and we want to show that $Z_h \simeq X_f \times_A Y_g$. So we want to construct a map:

$$\begin{aligned} (R_f)_0 \otimes_R (S_g)_0 &\xrightarrow{\simeq} ((R \otimes_A S)_h)_{(0,0)} \\ r/f^i \otimes s/g^j &\longmapsto ?? \end{aligned}$$

where $\deg r = \deg (f^i) = id$ and $\deg s = \deg (g^j) = je$. For some powers (written ?) we have:

$$h^? = \left(f^? \otimes g^?\right) \otimes f^i \otimes g^j \ ,$$

so we map this to:

$$?? = \frac{r \otimes s}{f^i \otimes g^j} = \frac{f^? r \otimes g^? s}{h^?}$$

and in the other direction we map

$$\frac{r \otimes s}{h} \to \frac{r}{f^d} \otimes \frac{s}{g^e} \ .$$

These details can be worked out without much trouble, and indeed

$$Z_h \cong X_f \times_A Y_g$$

so we are done. □

Consider again the situation that these are invertible sheaves. Then for $d > 0$, $e > 0$ we have

$$\operatorname{Proj}\left((R \otimes S)_{\mathbb{N}(d,e)} \right) \simeq X \times_A Y$$

but now $\mathcal{O}_Z(1)$ is $\mathcal{O}_X(d) \otimes \mathcal{O}_Y(e)$. So somehow we got the same scheme but with different line bundles.

3. Back to the Segré embedding

So the theorem tells us that $\operatorname{Proj}\left(k[\underline{x}] \otimes_k k[\underline{y}]\right)_\Delta = X \otimes_k Y$ but we still don't know exactly what the ideal is. So we somehow want to know the kernel of (1). So now we will take J generated by the proposed generators above, so this sits as follows:

$$J \subseteq I \subseteq k[\underline{z}] \longrightarrow \left(k[\underline{x}] \otimes_k k[\underline{y}]\right)_\Delta \ ,$$

and then the question is if $I = J$. Take monomials $\underline{x}^{\underline{a}}$ and $\underline{y}^{\underline{b}}$ where $|a| = |b| = d$ for a basis of

$$\left(k[\underline{x}] \otimes k[\underline{y}]\right)_\Delta \ .$$

We can always order monomials by writing them as

$$x_0^{a_0} \cdots x_l^{a_l} = x_{i_1} \cdots x_{i_d}$$

[1] If we wanted to be more efficient we would want something in degree $\operatorname{lcm}(d, e)$ but this works fine.

for $i_1 \leq i_2 \leq \cdots$ and similarly $\underline{y}^{\underline{b}} = y_{j_1} \cdots y_{j_d}$ for $j_1 \leq j_2 \leq \cdots$. Then there is an obvious map

$$z_{i,j} \cdots z_{i_d,j_d} \mapsto \underline{x}^{\underline{a}} \underline{y}^{\underline{b}} .$$

And now for $i < i'$ and $j > j'$ we have

$$z_{ij} z_{i'j'} = z_{i'j} z_{ij'}$$

so if we have a bad monomial we do moves such as this until it is ordered. Therefore such things span, so $J = I$, so those are really the equations of the Segré map.

The other issue is if X and Y themselves have equations. We can just insist on them in the obvious way, but we might worry if this is enough. Again from the theorem, this will be enough if these relations define the diagonal subring of the tensor product. But since tensor products preserve quotients we have

$$k\left[\underline{x}\right]/I \otimes_k k\left[\underline{y}\right]/J = k\left[\underline{x}, \underline{y}\right]/Ik\left[\underline{y}\right] + Jk\left[\underline{x}\right]$$

so this is sufficient.