LECTURE 17
MATH 256B

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We want to eventually talk about ample and very ample line bundles. These somehow give an embedding of a space into projective space. But today we will take a break from the technical machinery of Proj. In particular we will talk about how invertible sheaves are the same as line bundles and the general meaning of having a vector bundle.

1. Affine morphisms

Let $\varphi : X \to Y$ be a morphism of schemes. This is *affine* if $\varphi^{-1}(U)$ is affine for every affine open $U \subseteq Y$.

Example 1. If X is a vector bundle[1]over Y the projection will be an affine morphism.

Example 2. If φ is a closed embedding this is affine.

Warning 1. A locally closed embedding or an open embedding is not in general an affine morphism.

Example 3. The inclusion $X \hookrightarrow \mathbb{A}^2$ is not an affine morphism simply because $\mathbb{A}^2$ is affine and X is not.

1.1. **Locality.** This definition seems innocuous, but there is a subtle technical issue. In particular this notion is defined in a somehow local way. So now we might wonder if we can determined whether or not a morphism is affine based only on the preimages of an open covering of Y. One thing that is true is the following. Let $U = \operatorname{Spec} R \subseteq Y$ such that $\varphi^{-1}(U) = \operatorname{Spec} S$ is affine. Then U has a bunch of affines inside of it, the U_fs, and then $\varphi^{-1}(U_f) = \operatorname{Spec} S_f$ is affine as well. So as long as Y has a covering by affines with affine preimage, then there is a base of the topology consisting of affines which have affine preimages.

So the question is the following. Suppose $Y = \operatorname{Spec} R$ is affine such that we have a finite cover

$$Y = \bigcup_{i=1,\ldots,n} Y_{f_i}$$

and $\varphi^{-1}(Y_{f_i}) = X_{f_i}$ is affine for all i. Does this imply that X is affine? The answer is yes.

Proof. First notice that φ is quasi-compact and separated. As we have seen this means $\varphi_* \mathcal{O}_X = \mathcal{A}$ is a qco $\mathcal{O}_Y$-module (in fact it is a sheaf of $\mathcal{O}_Y$-algebras). Since

Date: March 4, 2019.

[1]We don't know what this means yet, but we will soon.

$Y = \operatorname{Spec} R$, we have that $\mathcal{A} = \tilde{A}$ where $A = \mathcal{O}_Y(A) = \mathcal{O}_X(X)$. In the same way, since $\mathcal{A}$ is now the sheaf associated to this module, we have

$$\mathcal{A}(Y_{f_i}) = A_{f_i} = \mathcal{O}_X(X_{f_i}) \ .$$

But we assumed X_{f_i} was affine, so $X_{f_i} = \operatorname{Spec} A_{f_i}$, and we want to show that $X = \operatorname{Spec} A$ is affine. From the ring homomorphism $A \to \mathcal{O}_X(X)$ we have a morphism $X \to \operatorname{Spec} A$, and we also have a morphism $X_{f_i} \to \operatorname{Spec} A_{f_i}$. In particular, the $\operatorname{Spec} A_{f_i}$ cover A, and the following diagram commutes:

$$\begin{array}{ccc} X & \longrightarrow & \operatorname{Spec} A \\ \uparrow & & \uparrow \\ X_{f_i} & \xrightarrow{\sim} & \operatorname{Spec} A_{f_i} \end{array}$$

so we do get an isomorphism $X \simeq \operatorname{Spec} A$. □

2. The $\underline{\operatorname{Spec}}$ construction

Let $\varphi : X \to Y$ be an affine morphism. So $\mathcal{A} = \varphi_* \mathcal{O}_X$ is a sheaf of $\mathcal{O}_X$ algebras. In particular, this is a qco sheaf as before. This sheaf basically tells us what X is. For $U = \operatorname{Spec} R \subseteq Y$ we have $\varphi^{-1}(U) = \operatorname{Spec} S$, and in particular

$$S = \mathcal{O}_X\left(\varphi^{-1}(U)\right) = \mathcal{A}(U) \ .$$

Now we have a morphism $\mathcal{O}_Y(U) = R \to \mathcal{A}(U) = S$, which gives us a map $\operatorname{Spec} S \to \operatorname{Spec} R$. So by some patching argument we can somehow recover X.

In particular, we didn't need X at all. We could have just started with any old qco sheaf. One obvious reason this needs to be qco is because otherwise we would get a qco sheaf of algebras back at the end. The actual reason is somehow hidden in the patching argument. This means affine schemes over Y, i.e. schemes with an affine morphism to Y, are in one-to-one equivalence to qco $\mathcal{O}_Y$-algebras $\mathcal{A}$. This is somehow a generalization of the story for affines. In the affine case going from algebras to schemes is just saying that ring homomorphisms correspond to morphisms of Spec. Motivated by this, for $\mathcal{A} = \varphi_*(\mathcal{O}_X)$ we write $X = \underline{\operatorname{Spec}}(\mathcal{A})$.

Remark 1. Even the universal property of Spec holds for this object as well. In particular for an arbitrary scheme over Y, we have that a Y-morphism to X is the same as a morphism of the associated algebras in the other direction:

$$\begin{array}{ccccc} Z & & \rightsquigarrow & & X \\ & \searrow & & \swarrow & \\ & & Y & & \end{array} .$$

Remark 2. If we are given a sheaf of graded algebras over Y we can do a similar thing to form some sort of $\underline{\operatorname{Proj}}$ in the analogous way.

2.1. Examples.

Example 4. For $i : X \hookrightarrow Y$ a closed embedding, and $U = \operatorname{Spec} R$ an affine in Y we have that $i^{-1}(U) = X \cap U = \operatorname{Spec} R/I$ is affine. Then $\mathcal{I}(U) \to R/I$ where $\mathcal{I} \subseteq \mathcal{O}_Y$ is a qco ideal sheaf. Then we can write $\mathcal{O}_Y/\mathcal{I} = i_*\mathcal{O}_X$ (is the $\mathcal{A}$ from above) and so we have $X = \underline{\operatorname{Spec}}(\mathcal{O}_Y/\mathcal{I})$.

3. Vector bundles

3.1. **Classical case.** Let $k = \overline{k}$ and let X be a classical variety. In this case a vector bundle is a classical algebraic variety V and a map $\varphi : V \to X$ where this locally looks like a product with a vector space. One example of such a thing is the trivial rank n vector bundle $X \times \mathbb{A}^n_k \to X$.

Definition 1. $\begin{array}{c} W \\ \downarrow \varphi \\ X \end{array}$ is a rank n vector bundle if we have an open cover $U \subseteq X$ such that $\varphi^{-1}(U) \cong \mathbb{A}^n_k \times_k U$ and given $U, V \subseteq X$ we have

$$\varphi^{-1}(U \cap V) \cong_u \mathbb{A}^n_k \times (U \cap V) \cong_v \mathbb{A}^n_k \times (U \cap V)$$

and then we insist that the map between these is k-linear on the fibers.

Now we want to somehow rephrase this in terms of functions. Consider generating sections $e_1, \cdots, e_n$ of $\varphi^{-1}(U)$ and $e'_1, \cdots, e'_n$ on $\varphi^{-1}(V)$. Then we have $\varphi^{-1}(U \cap V) = \mathbb{A}^n_k \times (U \cap V)$, and we want there to be $M \in \mathrm{GL}_n(U \cap V)$ such that

$$\begin{pmatrix} e'_1 \\ \cdots \\ e'_n \end{pmatrix} = M \begin{pmatrix} e_1 \\ \cdots \\ e_n \end{pmatrix} .$$

As it turns out this is somehow a more correct version of this data.

Definition 2. $\begin{array}{c} W \\ \downarrow \varphi \\ X \end{array}$ is a rank n vector bundle if φ is affine, the sheaf of algebras $\varphi_* \mathcal{O}_W$ is locally of the form $\mathbb{Z}[x_1, \cdots, x_n] \otimes_{\mathbb{Z}} \mathcal{O}_X$ on open sets U covering X, and if we consider the two isomorphisms coming from U and V:

$$\mathbb{Z}[\underline{x}] \otimes \mathcal{O}_{U \cap V} \simeq \varphi_* \mathcal{O}_{U \cap V} \simeq \mathbb{Z}[\underline{y}] \otimes \mathcal{O}_{U \cap V}$$

then on $Y \subseteq U \cap V$ we require that the resulting composition is a linear isomorphism

$$\mathcal{O}(Y)[\underline{x}] \cong \mathcal{O}(Y)[\underline{y}] .$$

I.e. this is determined by

$$\begin{pmatrix} y_1 \\ \cdots \\ y_n \end{pmatrix} = M \begin{pmatrix} x_1 \\ \cdots \\ x_n \end{pmatrix}$$

for $M \in \mathrm{GL}_n(\mathcal{O}(Y))$.

Notice that from this description it is immediate that an invertible sheaf is the same as a rank 1 vector bundle.