LECTURE 18
MATH 256B

LECTURE: PROFESSOR MARK HAIMAN
NOTES: JACKSON VAN DYKE

Recall affine n-space over a scheme X, $\mathbb{A}^n_X$, is defined as follows. First consider the qco sheaf of $\mathcal{O}_X$-algebras $\mathcal{O}_X[t_1, \cdots, t_n]$. Then

$$\mathbb{A}^n_X = \underline{\operatorname{Spec}}\left(\mathcal{O}_X[t_1, \cdots, t_n]\right) .$$

If $X = \operatorname{Spec} R$ is affine itself, this is just $\mathbb{A}^n_R = \operatorname{Spec}(R[t_1, \cdots, t_n])$.

1. Vector bundles

We want to give $\mathbb{A}^n_X$ the structure of the 'trivial rank n vector bundle over X.' To give an X-morphism from $\mathbb{A}^n_X \to \mathbb{A}^n_X$ is the same as giving an $\mathcal{O}_X$-algebra homomorphism $\mathcal{O}_X[t_1, \cdots, t_n] \to \mathcal{O}_X[t_1, \cdots, t_n]$.

Remark 1. We know $T \to \operatorname{Spec} R$ corresponds to $\mathcal{O}_T(T) \leftarrow R$ so a map

$$\begin{tikzcd} T \arrow[rr, "f"] \arrow[dr, "q"'] & & \operatorname{Spec} R \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

corresponds to a $\mathcal{O}_X$-algebra homomorphism $R \to q_* \mathcal{O}_T$, or equivalently $\mathcal{O}_T$-algebra morphisms $q^* R \to \mathcal{O}_T$. The second way is kind of preferable because these morphisms are always of qco sheaves, whereas $q_* \mathcal{O}_T$ is only qco if we assume something about T.

Then the point is that the endomorphisms of $\mathbb{A}^n_X$ we allow are ones which correspond to linear changes of variables on the underlying algebras, i.e. maps which send $t_i \mapsto \sum a_{ij} t_i$. Equivalently we have

$$\operatorname{Aut}_{\mathbf{Vect}/X}\left(\mathbb{A}^n_X\right) = \mathrm{GL}_n\left(\mathcal{O}_X(X)\right) .$$

With this way of thinking about it, we say:

Definition 1. A general geometric vector bundle should be a scheme E with a morphism $p : E \to X$ such that X is covered by opens U such that $p^{-1}(U) \simeq \mathbb{A}^n_U \to U$ such that the composition

$$\mathbb{A}^n_{U \cap V} \xrightarrow{\simeq_U} p^{-1}(U \cap V) \xrightarrow{\simeq_V} \mathbb{A}^n_{U \cap V}$$

is in $\mathrm{GL}_n(\mathcal{O}(U \cap V))$.

Date: March 6, 2019.

1.1. **Abelian group scheme.** We claim that $\mathbb{A}^n_X$ is in fact an abelian group scheme over X. To see this, we need to specify a product $\mathbb{A}^n_X \times \mathbb{A}^n_X \to \mathbb{A}^n_X$. We can specify this with a map

$$\mathcal{O}_X[\underline{t}] \to \mathcal{O}_X[\underline{t}] \otimes_{\mathcal{O}_X} \mathcal{O}_X[\underline{t}] = \mathcal{O}_X[u_1, \cdots, u_n, v_1, \cdots, v_n]$$

which sends $t_i \mapsto u_i + v_i$.

As always with group schemes, it is better to think of the functor represented by this as being group valued. Maps f as in:

$$\begin{tikzcd} T \arrow[rr, "f"] \arrow[dr, "q"'] & & \mathbb{A}^n_X \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

correspond to $\mathcal{O}_T$-algebra homomorphisms

$$\mathcal{O}_T[t_1, \cdots, t_n] \to \mathcal{O}_T$$

which correspond to $(s_1, \cdots, s_n) \in \mathcal{O}_T(T)^n$, so this is indeed a group. In fact we can also multiply them so $\mathbb{A}^n_X$ is a group scheme, and $\mathbb{A}^1_X$ is a (commutative unital) ring scheme because the functor that it represents on schemes over X is just

$$\underline{\underline{\mathbb{A}^1_X}}(T) = \mathcal{O}_T(T) \ .$$

Therefore $\mathbb{A}^n_X$ is in fact an $\mathbb{A}^1_X$-module scheme.

Definition 2. A geometric rank n vector bundle $p : E \to X$ is an $\mathbb{A}^1_X$-module scheme such that locally

$$\begin{tikzcd} p^{-1}(U) \arrow[r] \arrow[d, "\cong"] & U \\ \mathbb{A}^n_U \arrow[ur] & \end{tikzcd}$$

as an $\mathbb{A}^1_U$-module scheme.

With some work, we can see that this is the same as the first definition, which was too abstract, but this is kind of too complicated, so we have a third one.

1.2. **Invertible sheaves.** Let $R = \mathcal{O}_X[t_1, \cdots, t_n]$ be a sheaf of graded $\mathcal{O}_X$-algebras. Then let $p : E = \underline{\mathrm{Spec}}(\mathcal{A}) \to X$ be a vector bundle in the first sense. Note $\mathcal{A}$ is graded here. We know $\mathcal{A}_0 = \mathcal{O}_X$, and $\mathcal{A}_1 = \mathcal{E}$ is an $\mathcal{O}_X$-module free of rank n, i.e. it is isomorphic to $\mathcal{O}^n_X$. The rest of this is the symmetric algebra over $\mathcal{O}_X$:

$$\mathcal{A} = S_{\mathcal{O}_X}(\mathcal{A}_1)$$

since this is just what it means to be locally isomorphic to a polynomial ring.

So given a vector bundle, we get a locally free sheaf of rank n, but this is reversible too. So invertible sheaves $\mathcal{L}$ correspond to rank 1 vector bundles L, i.e. line bundles. The idea is that $\mathcal{E}$ is somehow dual to E. Given a map $Z \to X$, we can always get a sheaf of sections σ_Z, which assigns the sections over U to U:

$$\sigma_Z(U) = \left\{ \begin{tikzcd} U \arrow[r] \arrow[dr, "1"] & p^{-1}(U) \arrow[d, "p"] \\ & U \end{tikzcd} \right\} = \underline{\underline{Z}}(U) \ .$$

In the case of a vector bundle, $\underline{\underline{E}}(U)$ takes values in $\mathbb{A}_1$-modules, and of course $\underline{\underline{\mathbb{A}_X^1}}(U) = U$, so $\sigma_{\mathbb{A}_X^1} \simeq \mathcal{O}_X$. Then globally we have that the elements of $\underline{\underline{E}}(U)$ are determined by maps $\mathcal{E}(U) \to \mathcal{O}_U(U)$, i.e. linear functionals. We write:

$$\underline{\underline{E}} = \mathcal{E}^{\vee} \ .$$

The general idea is as follows. Consider some space X and two $\mathcal{O}_X$ modules $\mathcal{M}$ and $\mathcal{N}$. Then we can define a presheaf

$$\operatorname{Hom}(\mathcal{M}, \mathcal{N})(U) = \operatorname{Hom}_{\mathcal{O}_U}\left(\mathcal{M}|_{UI}, \mathcal{N}|_U\right)$$

which is in fact a sheaf. If X is a scheme and these are qco sheaves this might not be qco. We would need some other conditions for this to be the case. But if $\mathcal{M}$ is locally free, then this will be qco and in fact locally free as well since $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{N}) \simeq \mathcal{N}$. Then

$$\mathcal{E}^{\vee} = \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$$

is locally free of rank n.