LECTURE 19
MATH 256B

LECTURE: PROFESSOR MARK HAIMAN
NOTES: JACKSON VAN DYKE

Today we will wrap up our discussion of how a geometric vector bundle has a sheaf of sections which is dual to the locally free sheaf associated with it.

1. Geometric vector bundles

1.1. Hom functor. Consider some ringed space X equipped with $\mathcal{O}_X$ and consider two $\mathcal{O}_X$ modules $\mathcal{M}$ and $\mathcal{N}$. Then we define a presheaf of $\mathcal{O}_X$ modules

$$\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{M},\mathcal{N})(U) = \operatorname{Hom}_{\mathcal{O}_U}\left(\mathcal{M}|_U, \mathcal{N}|_U\right)$$

which is actually a sheaf, so we don't even need to sheafify it. Note that Hom is left exact in $\mathcal{N}$, and also left exact in $\mathcal{M}$, but we should think of $\mathcal{M} \in (\mathcal{O}_X\text{-}\mathbf{Mod})^{\mathrm{op}}$.

Remark 1. Recall that it's somehow a general philosophy that things which are left exact and right adjoint (like Hom) somehow play nice with sections, meaning they don't need to be sheafified, and things which are right exact and left adjoint (like $\otimes$) typically need to be sheafified. The opposite is true for stalks, i.e. in general we have a map

$$\operatorname{Hom}(\mathcal{M},\mathcal{N})_p \to \operatorname{Hom}_{\mathcal{O}_p}(\mathcal{M}_p,\mathcal{N}_p)$$

but it isn't generally an isomorphism.

Note that as we might expect,

$$\operatorname{Hom}(\mathcal{O}_X,\mathcal{N}) = \mathcal{N} .$$

1.2. Locally presented. Now assume we have a local presentation

$$\mathcal{O}^{(J)} \to \mathcal{O}^{(I)} \to \mathcal{M} \to 0$$

on U. Since Hom is left exact we can apply $\operatorname{Hom}(-,\mathcal{N})$ to get the following exact sequence:

$$0 \to \operatorname{Hom}(\mathcal{M},\mathcal{N}) \to \mathcal{N}^I \to \mathcal{N}^J .$$

If X is a scheme, $\mathcal{M}$ is locally finitely presented (which implies qco) and $\mathcal{N}$ is qco, then $\operatorname{Hom}(\mathcal{M},\mathcal{N})$ is qco since it is the kernel of a map of qco sheaves (these are qco because the presentation was finite).

In particular, if $X = \operatorname{Spec} R$, then

$$\operatorname{Hom}\left(\tilde{M},\tilde{N}\right) = \operatorname{Hom}_R(M,N)^{\sim}$$

where M is finitely presented.

Date: March 8, 2019.

1.3. **Locally free.** Let $\mathcal{E}$ be locally free of finite rank n. Then we define

$$\mathcal{E}^\vee := \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$$

which is locally free of rank n as well. Of course there is always a canonical evaluation map

$$\mathcal{M} \to \operatorname{Hom}(\operatorname{Hom}(\mathcal{M}, \mathcal{N}), \mathcal{N}) ,$$

but if $\mathcal{M} = \mathcal{E}$ and $\mathcal{N} = \mathcal{O}$, then this is an isomorphism

$$\mathcal{E} \xrightarrow{\simeq} (\mathcal{E}^\vee)^\vee .$$

Note that if $\mathcal{L}$ is invertible, so locally free of rank 1, then $\mathcal{L}^\vee$ is also invertible. In particular it is as follows. We always have a map

$$\mathcal{M} \otimes \operatorname{Hom}(\mathcal{M}, \mathcal{N}) \to \mathcal{N}$$

which in this case is

$$\mathcal{E} \otimes \mathcal{E}^\vee \to \mathcal{O}$$

and in particular, if $\mathcal{E}$ is invertible this is an isomorphism, so $\mathcal{L}^\vee = \mathcal{L}^{\otimes -1}$ is the tensor inverse.

1.4. **Adjoint relationship.** Note that the connection between these things is a bit more general. In particular we have an adjoint relationship. If we fix $\mathcal{M}$ then $\operatorname{Hom}(\mathcal{M}, -)$ is right adjoint to $\mathcal{M} \otimes -$. I.e. this says that for sheaves $\mathcal{A}$ and $\mathcal{B}$ we have the correspondence:

$$\mathcal{A} \to \operatorname{Hom}(\mathcal{M}, \mathcal{B}) \qquad \leftrightsquigarrow \qquad \mathcal{M} \otimes \mathcal{A} \to \mathcal{B} .$$

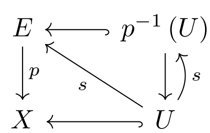
This all comes from the universal property of $\otimes$ and the definition of Hom. We can even jazz this up a little more to get that

$$\operatorname{Hom}(\mathcal{A}, \operatorname{Hom}(\mathcal{M}, \mathcal{B})) \cong \operatorname{Hom}(\mathcal{M} \otimes \mathcal{A}, \mathcal{B}) .$$

1.5. **Vector bundles.**

Claim 1. If $E = \operatorname{Spec} S(\mathcal{E}) \to X$, then $\sigma_E(U) = \underline{\underline{E}}(U)$ has $\sigma_E \cong \mathcal{E}^\vee$.

The diagram here is:



.

$x \in \mathcal{E}(U)$ is a "function" on $p^{-1}(U)$ in the sense that it is in $\mathcal{O}_E\left(p^{-1}(U)\right)$. And then we can "compose" with s to get an element of $\mathcal{O}_X(U)$.

By the adjointness above, the following are interchangeable:

$$\sigma_E \otimes \mathcal{E} \to \mathcal{O} \qquad \sigma_E \to \mathcal{E}^\vee \qquad \mathcal{E} \to \sigma_E^\vee$$

so we just need to check that the middle is an isomorphism as long as it is locally free. To see this let's look at the case where it's actually free. So let $\mathcal{E} \cong \mathcal{O}^n$. Then

$$S(\mathcal{E}) = \mathcal{O}[x_1, \cdots, x_n]$$

and

$$E = \mathbb{A}^n_X = \operatorname{Spec} R[\underline{x}] \to X = \operatorname{Spec} R .$$

Then

$$V(x_i - 1, x_j \mid j \neq i) \simeq \operatorname{Spec} R$$

so this is an isomorphism.

1.6. **An example.** Let $X = \mathbb{P}^n_k$ be classical projective space. In particular we want to think of this as the set of 1 dimensional $L \subseteq k^{n+1}$. Geometrically, X has a tautological line bundle. One way to think of this is that there is a map $\mathbb{A}^{n+1} \setminus 0 \to \mathbb{P}^n_k$. The fibers are the lines we want but missing a point. This is fine, because there's a map $\mathbb{A}^{n+1} \setminus 0 \xrightarrow{j} \mathbb{A}^{n+1}_k$ so we have a map $f : \mathbb{A}^{n+1} \setminus 0 \to \mathbb{P}^n_k \times \mathbb{A}^{n+1}_k$ and then L is the closure of $\operatorname{im}(f)$.

On $\mathbb{P}^n_k$ we also have $\mathcal{L} = \mathcal{O}(1)$. The relationship between these is that

$$L = \operatorname{Spec} S(\mathcal{O}(1))$$

and

$$\sigma_L = \mathcal{L}^\vee = \mathcal{O}(-1) \ .$$

The reason is as follows. Locally we want some pairing between elements of $\mathcal{O}(1)$ and sections of L. Basically, to see that this is the right correspondence it is enough to do it for a generating section on affines. The global sections of $\mathcal{O}(1)$ are

$$\Gamma(\mathbb{P}^n_k, \mathcal{O}(1)) = k[x_0, \cdots, x_n]_1 \ .$$

And then $f = \sum a_i x_i$ for $a_i \in k$ is a function on $\mathbb{A}^{n+1}_k$. The idea is that we can evaluate x_i on a section $s \in \sigma_L$, and then $x_i(s)$ can be thought of as a function.

As long as $n > 0$, this has no nontrivial global sections. Of course we can find nontrivial local sections such that as $x_i = 1$ and then we can take all the lines such that $x_i = 0$ and these meet at exactly one point. But we can't do this globally besides the zero section.

2. Back to Proj

This is a little preview of what we will do next time. Consider a map $X \to Y = \operatorname{Proj} R$. If R is sufficiently nice, then $\mathcal{O}(1)$ will be an invertible sheaf. Even if not, it might be invertible on part of it. And then it might happen that there is a line bundle $\mathcal{L}$ on X given by $\varphi^*\mathcal{O}(1)$.

Then we can ask if this can be reversed given X and $\mathcal{L}$. And we saw that $\Gamma_+(\mathcal{L}) = \bigoplus_d \mathcal{L}^{\otimes d}(X)$ sort of does this. This might be an ugly ring, but at least on the image of X there are somehow enough invertible $\mathcal{O}(d)$s. Then we can ask when $\mathcal{L}$ gives an embedding of X as a locally closed subscheme. Basically this is the notion of $\mathcal{L}$ being ample. As it turns out there is a very beautiful theory of this.