LECTURE 2
MATH 256A

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1. Proj construction

Recall that last time we did the following. Let R be an $\mathbb{N}$-graded ring, and define the irrelevant ideal:

$$R_+ = \bigoplus_{d>0} R_d \ .$$

Recall we defined this because $V(R_+) \subseteq \operatorname{Spec} R$ is the '$\mathbb{G}_m$ fixed locus.' Then we define

$$\operatorname{Proj} R = \left\{ P \in \operatorname{Spec} R \setminus V\left(R^+\right) \mid P \text{ is graded,} \right\} \ .$$

These are the generic points of the collection of $\mathbb{G}_m$ invariant irreducible closed subvarieties of $\operatorname{Spec} R$ which are not $\mathbb{G}_m$ fixed.

1.1. Scheme structure.

Now we need to produce a sheaf of functions to make it a scheme. The idea here is that on some prime P, we look at R_P, but we want the degree 0 part. But there is an issue, which is that R_P isn't graded in general. One ad hoc way is to just invert the homogeneous elements, and then it is graded so we can just take the 0 degree part. But this is sort of poorly motivated, so we will do the following. First of all, R_+ is by definition generated by homogeneous elements of positive degree. If $Y = \operatorname{Spec} R$, then $Y \setminus V(R_+)$ is covered by the basic opens $Y_f = \operatorname{Spec} R\left[f^{-1}\right] = \operatorname{Spec} S$ for $f \in R_d$ where $d > 0$. Note that S is now a $\mathbb{Z}$ graded ring since f^{-1} has negative degree. Also note that $X_f = Y_f \cap X$ consists of the graded prime ideals P in R_f. Then we have some degree 0 part $S_0 \subseteq S$, so of course $\operatorname{Spec} S \to \operatorname{Spec} S_0$. And in fact,

$$X_F \hookrightarrow Y_f = \operatorname{Spec} S \to \operatorname{Spec} S_0 \ .$$

This map just takes a graded prime, and maps it to its degree 0 part. We claim that this composition is a homeomorphism.

Let $f \in R_d$, then for $S = R\left[f^{-1}\right]$ we can consider the subring $S_{(\mathbb{Z}d)}$ of elements with degree a multiple of d. Each $S_{nd} \simeq S_0$ as an S_0 modules. In particular, $S_{(\mathbb{Z}d)} \simeq S_0\left[f^{\pm 1}\right]$ is just the Laurent polynomial ring. Then there is a correspondence between graded ideals in $S_{\mathbb{Z}d}$ and arbitrary ideals in S_0. The correspondence is just $J \mapsto J_0$ and $I \mapsto IS_{\mathbb{Z}d}$. Note that this preserves prime ideals, since $S_{\mathbb{Z}d}/J \simeq (S_0/I)\left[f^{\pm 1}\right]$ is also a Laurent polynomial ring, and a Laurent polynomial ring is an integral domain iff the ring is.

Now we show it is a homeomorphism. If we take $V(I) \subseteq \operatorname{Spec} S_0$ on the side consisting of ideals in S_0, then this corresponds to $V(IS_{\mathbb{Z}d})$ which is also closed.

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1.2. **Thinning out a general ring.** We want this to work in S, and not just $S_{\mathbb{Z}d}$. So we want to prove that for any graded ring, if we thin it out like this, then Spec of graded ideals of this subring is homeomorphic to Spec of the ring. So now we have the set X_f of graded primes in $\operatorname{Spec} S$, and this maps to the graded primes $P \in \operatorname{Spec} S_{\mathbb{Z}d}$ just by taking the components with degree a multiple of d. Then to reverse this, for any Q a graded prime in $\operatorname{Spec} S_{\mathbb{Z}d}$ we look at the homogeneous elements and see if their dth power is in Q:

$$Q \mapsto P_n = \left\{ g \in S_n \mid g^d \in Q_{nd} \right\} \ .$$

Lemma 1. *In any graded ring S, a collection of abelian subgroups $I_n \subseteq S_n$ are components of an ideal iff $S_m I_n \subseteq I_{m+n}$.*

Now let's see that this works here. For g in S_n and $h \in S_m$, then gh is again homogeneous, and indeed $(gh)^d = g^d h^d$ is in $Q_{(n+m)d}$ and so in P_{n+m}.

Lemma 2. *A graded ideal I is prime if the product of any nonzero homogeneous elements is nonzero.*

For g and h homogeneous, since $gh \in P$, then $g^d h^d \in Q$, so either g or h is in Q, so P is prime.

1.3. **Putting it together.** Putting these together, we get that $X_f \to \operatorname{Spec} S_0$ where $S_0 = R\left[f^{-1}\right]_0$ is a homeomorphism. Then we can just equip X_f with the structure sheaf of this. This gives $\mathcal{O}_{X_f}$ and we want it to be $\mathcal{O}_X|_{X_f}$. So let's look at a point $p \in X_f$ and look at what the ring actually is:

$$\mathcal{O}_{X_f,p} = \left(R\left[f^{-1}\right]_0\right)_{(P_f)_0}$$

which has generic elements which look like a/f^n where a is not in P, and has degree $-nd$. But since f is already inverted, this is just inverting these a. So define the (multiplicative) set

$$T = \bigcup_n R_n \setminus P_n \ .$$

Then we claim that $T^{-1}R$, as a localization of $R\left[f^{-1}\right]$, is graded since we inverted only homogeneous things, and when we take $\left(T^{-1}R\right)_0$ this is exactly $\mathcal{O}_{X_f,P}$ which only depends on P, not f. So we did indeed get the ad hoc construction from the beginning. From this point of view we see that these are consistent regardless of which f we have.

Note that for $Y = \operatorname{Spec} R$ and $Y_f = \operatorname{Spec} R\left[f^{-1}\right]$ for $f \in R_d$ for $d > 0$ we should geometrically think of $X_f = \operatorname{Spec} R\left[f^{-1}\right]$ as a sort of local quotient by the $\mathbb{G}_m$ action.

1.4. **Examples.**

Example 1. Let $R = k\left[x_0, \cdots, x_n\right]$ with the usual grading, and let $X = \operatorname{Proj} R$. Then the X_{x_i} cover X because the x_i generate R_+. These are

$$X_{x_i} = \operatorname{Spec} k\left[x_0, \cdots, x_n, x_i^{-1}\right]_0 = \operatorname{Spec} k\left[x_0/x_i, \cdots, x_n/x_i\right] = \mathbb{A}^n$$

since we can omit $x_i/x_i = 1$. So $\operatorname{Proj} R = \mathbb{P}^n$, and this is the same as the sort of ad hoc way we look at this before. In particular as long as $k = \overline{k}$ then this is classical projective space from before.