LECTURE 21
MATH 256B

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Recall we have a scheme X and line bundle (invertible sheaf) $\mathcal{L}$. Then canonically we get an open set $W \subseteq X$ and a map $\varphi : W \to \operatorname{Proj} \Gamma_+ (\mathcal{L})$.

Proposition 1. *$W = X$ and φ is a locally closed embedding iff there exist enough $s \in \mathcal{L}^{\otimes d}(X)$ such that*

(i) X_s cover X,
(ii) X_s is affine, and
(iii) for all $g \in \mathcal{O}(X_s)$ there exists n such that $gs^{\otimes n}$ extends to $t \in \mathcal{L}^{\otimes nd}(X)$. Note $s^{\otimes n} \in \mathcal{L}^{\otimes nd}(X_s)$.

If we assume qco we can weaken these assumptions. If we have $U \subseteq X$ such that $\mathcal{L}|_U \cong \mathcal{O}_U$ then $U \cap X_s = U_s$ is affine. In addition, if $X_s \subseteq U$ then X_s is affine. In particular, if the X_ss form a base of the topology on X then we get (i) and (ii) from proposition 1 for free. In addition, this implies $X_s \cap X_t = X_{st}$ is affine and X is quasi-separated.

Now suppose $\varphi : X = W \hookrightarrow Y$ is a topological embedding,[1] i.e. a homeomorphism onto a subspace of Y. So $Y = \operatorname{Proj} \Gamma_+ (\mathcal{L})$, and as always the Y_ss form a base of the topology on Y. This implies that the X_ss form a base for the topology on X. So it is certainly necessary that the X_ss form a base. As such, in this case we only need to worry about (iii).

To see this, assume X is quasi-compact and quasi-separated. The idea is that this gives us (iii) for free now. Recall that in this case if $\mathcal{M} \in \mathbf{QCoh}(X)$, and $f \in \mathcal{O}(X)$, then this implies $\mathcal{M}(X_f) = \mathcal{M}(X)_f$.

Also recall that if $p : X \to T$ is quasicompact and quasi-separated, then this means $p_* (\mathbf{QCoh}(X)) \subseteq \mathbf{QCoh}(T)$. In particular, for $T = \operatorname{Spec} A$ and $M = \mathcal{M}(X)$ we want to know if $p_* (\operatorname{Spec} A_f) = M_f$ and we know we have

$$p_* (\operatorname{Spec} A_f) = \mathcal{M}(X_f) \ .$$

So the question is if $\mathcal{M}(X)_f = \mathcal{M}(X_f)$. The way we saw this was covering X with affines U_i such that $\mathcal{M}|_{U_i} = \tilde{M}_i$. Then $U_i \cap U_j = \cup U_{ijk}$. Then we compare the two exact sequences:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{M}(X) & \longrightarrow & \bigoplus_i \mathcal{M}(U_i) & \longrightarrow & \bigoplus_{ijk} \mathcal{M}(U_{ijk}) \\ & & & & \downarrow{\scriptstyle (-)_f} & & \downarrow{\scriptstyle (-)_f} \\ 0 & \longrightarrow & \mathcal{M}(X_f) & \longrightarrow & \bigoplus_i \mathcal{M}\left((U_i)_f\right) & \longrightarrow & \bigoplus_{ijk} \mathcal{M}\left((U_{ijk})_f\right) \end{array}$$

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[1] Note that importantly we are not assuming this is an embedding of schemes

which tells us that

$$\mathcal{M}\left(X_f\right) = \mathcal{M}\left(X\right)_f \ .$$

This argument is very fundamental. We have seen it in the proof showing that qco sheaves are sheaves associated to modules on schemes, and a second time as above when we saw that quasi-compact and quasi-separated morphisms preserve qco sheaves. This argument generalizes in the following way. For $f \in \mathcal{L}\left(X\right)$ we have

$$\mathcal{M}\left(X_f\right) = \Gamma_+\left(X, \mathcal{M}, \mathcal{L}\right) = \bigoplus_{d \geq 0} \mathcal{M} \otimes \mathcal{L}^{\otimes d}\left(X\right) \ .$$

Then cover X with affine U_i such that $\mathcal{L}|_{U_i} \simeq \mathcal{O}_{U_i}$. Then we have effectively the same sequence as above in the second line, and we write the first line for $\bigoplus_d \mathcal{M} \otimes \mathcal{L}^{\otimes d}$.

The upshot is that (iii) is always satisfied for X quasicompact and quasiseparated. So now just assume X is quasi-compact and also that there is a topological embedding given by this canonical map. Then as we said before the X_fs will be a base of the topology, so the affine ones will be a base, and then we get quasi-separated for free and therefore (iii) for free. Therefore we have the following:

Theorem 1. *Let X be quasicompact, and $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Then TFAE:*

(a) $\varphi : X = W \hookrightarrow Y = \operatorname{Proj}\Gamma_+\left(\mathcal{L}\right)$ is a locally closed embedding.
(b) $X = W$ and φ is a topological embedding.
(c) The X_ss form a base of the topology on X.
(d) The affine X_ss cover X.

Proof. The more clear implications are $(a) \implies (b) \implies (c) \implies (d)$. Along the way we noticed that (c) implies that X is quasi-separated. In particular we saw that $(d) \implies (a)$ so we are done. □

Definition 1. $\mathcal{L}$ is *ample*[2] if X is quasicompact and $(a) - (d)$ from theorem 1 hold.

Definition 2. Let $f : X \to T$. Then $\mathcal{L}$ is *ample for f* if f is quasi-compact and T can be covered by affines U such that $\mathcal{L}$ is ample on $f^{-1}\left(U\right)$.

Remark 1. We are on our way to defining a quasi-projective morphism and quasi-projective variety.

So now assume we have f as in the definition. Then we have a sort of f-relative version of Γ_+:

$$f_+\mathcal{L} := \bigoplus_{d \geq 0} f_*\mathcal{L}^{\otimes d}$$

which is a quasi-coherent sheaf of graded $\mathcal{O}_T$-algebras. Then for $U = \operatorname{Spec} A \subseteq T$ we have

$$f_+\mathcal{L}\left(U\right) = \Gamma_+\left(f^{-1}\left(U\right), \mathcal{L}\right) \ .$$

Now we can piece these together to get:

$$X \longrightarrow T \longleftarrow \underline{\operatorname{Proj}}\left(f_+\left(\mathcal{L}\right)\right)$$

[2]Note this is a property which somehow X and $\mathcal{L}$ have, not just $\mathcal{L}$.

where the quasi-compact and quasi-separated gives us a canonical map $X \supset W \to \underline{\mathrm{Proj}}\,(f_+(\mathcal{L}))$. So $X = W \hookrightarrow \underline{\mathrm{Proj}}\,(f_+(\mathcal{L}))$ is a locally closed embedding.

Example 1. Let $X = \mathrm{Proj}\, R$ where $V(R_1) = \emptyset \subset X$. $\mathcal{L} = \mathcal{O}(1)$ is invertible. Recall $R_d \to \mathcal{L}^{\otimes d}$. This implies that $\mathcal{L}$ is (very) ample (if X is quasicompact).

Example 2. Let $\mathcal{L} = \mathcal{O}$. First of all X has to be quasicompact. Since $\mathcal{L}^{\otimes n} = \mathcal{O}$ we have

$$\Gamma_+(\mathcal{L}) = \mathcal{O}(X)[t]$$

and $Y = \mathrm{Spec}\,\mathcal{O}(X)$. In this case this means X is a quasicompact locally closed subset of an affine scheme. Therefore it is affine, so this is the same as saying X is a compact open subset of an affine scheme. Such things are called quasi-affine which is sufficient for $\mathcal{O}$ to be ample.