LECTURE 22
MATH 256B

LECTURE: PROFESSOR MARK HAIMAN
NOTES: JACKSON VAN DYKE

We will continue with some basic properties and examples of ample line bundles.

1. Ample line bundles

1.1. Definition review. Recall that, first of all, we really only want to deal with this concept on a quasi-compact scheme. We defined an invertible sheaf $\mathcal{L}$ to be *ample* if (equivalently)

(1) $\mathcal{L}$ gives $X \hookrightarrow \operatorname{Proj} \Gamma_+ (\mathcal{L})$,
(2) there exist enough $s \in \mathcal{L}^{\otimes d} (X)$ such that the X_s form a base of the topology, or
(3) there exist enough $s \in \mathcal{L}^{\otimes d} (X)$ such that the X_s form an affine covering.

Then we had a relative version, where we said that $\mathcal{L}$ is ample for $f : X \to T$ if f is quasicompact, and $\mathcal{L}$ is ample on $f^{-1} (U)$ for $U \subseteq T$ affine.

1.2. Some basic properties. An initial property of these is that $\mathcal{L}$ is ample iff $\mathcal{L}^{\otimes n}$ is ample.

Definition 1. $\mathcal{L}$ is *flexible* if the X_ss for $s \in \mathcal{L}^{\otimes d}$ cover X. Note that if X is quasi-compact, then this just means some $\mathcal{L}^{\otimes n}$ is globally generated.

Fact 1. *If we tensor an ample line bundle with a flexible line bundle this is still ample.*

This fact is most useful when we want to tensor an ample bundle with a globally generated line bundle. So this tells us this is in fact still ample.

Fact 2. *Let $\mathcal{L}$ be ample. If $M \in \mathbf{QCoh} (X)$ then*

$$\mathcal{M} (X_s) = (\Gamma_+ (\mathcal{M}, \mathcal{L})_s)_0$$

where

$$\Gamma_+ (\mathcal{M}, \mathcal{L}) = \bigoplus_d \mathcal{M} \otimes \mathcal{L}^{\otimes d} (X) \ .$$

The point is that somehow $\mathcal{M} = j^* \Gamma_+ (\mathcal{M}, \mathcal{L})^{\sim}$ where j is the embedding into the Proj.

For $a \in \mathcal{M} (X_s)$ we have that

$$a \otimes s^{\otimes n} \in \mathcal{M} \otimes \mathcal{L}^{\otimes nd} (X_s)$$

extends $t \in \mathcal{M} \otimes \mathcal{L}^{\otimes nd} (X)$.

Fact 3. *If $\mathcal{M}$ is locally finitely generated, then for some $n \gg 0$, $\mathcal{M} \otimes \mathcal{L}^{\otimes n}$ is generated by finitely many global sections.*

Date: March 15, 2019.

The point of this is that the following sheaf homomorphism is surjective

$$\mathcal{O}^m \twoheadrightarrow \mathcal{M} \otimes \mathcal{L}^{\otimes n} \ .$$

Then we can tensor with $\mathcal{L}^{\otimes -n}$ to get

$$\left(\mathcal{L}^{\otimes -n}\right)^m \twoheadrightarrow \mathcal{M} \ .$$

In particular, if everything is affine then both $\left(\mathcal{L}^{\otimes -n}\right)^m$ and $\mathcal{M}$ are locally finitely generated, and the kernel is finitely generated as well (at least in the Noetherian situation) so we get a presentation of $\mathcal{M}$.[1]

Suppose $\mathcal{M}$ is an invertible sheaf itself, and that there exists an ample bundle $\mathcal{L}$ on X. Then we know for some n, $\mathcal{M} \otimes \mathcal{L}^{\otimes n}$ is generated by global sections, which means we have that $\mathcal{L}' \coloneqq \mathcal{M} \otimes \mathcal{L}^{n+1}$ is ample and

$$\mathcal{M} = \mathcal{L}' \otimes \left(\mathcal{L}^{\otimes n+1}\right)^{\otimes -1} = (\text{ample}) \otimes (\text{ample})^{\otimes -1} \ .$$

The idea is somehow that we should think of this as telling us that there is some sort of ample cone inside of $\operatorname{Pic}(X)$. So somehow the Picard group is much more structured in this case. We somehow get a partial order from this.

1.3. **Examples.**

Example 1. Consider the flag variety G/B for $G = \mathrm{GL}_n(\mathbb{C})$. Recall a general flag consists of choices F_k (of dimension k) such that

$$F_0 = 0 \subset F_1 \subset \cdots \subset F_{n-1} \subset \mathbb{F}_n = \mathbb{C}^n \ .$$

A particular Borel subgroup B is given by upper triangular matrices. Now we can consider a chain of subspaces

$$0 \subset \mathbb{C}^1 = \mathbb{C} \cdot e_1 \subset \cdots \subset \mathbb{C}^{n-1} \subset \mathbb{C}^n$$

which is stabilized by B. Therefore the set of all such flags is identified with G/B. This is an algebraic variety because if we just look at subspaces at one level, this is the Grassmannian variety, and then the whole thing is a product of these.

For a general flag F_*, we have $F_k/F_{k-1} \cong \mathbb{C}$ so we get a natural line bundle

$$\begin{array}{c} L_k \\ \downarrow \\ G/B \end{array} \ .$$

Now it turns out

$$L_n \otimes \cdots \otimes L_{n-k+1} = \Lambda_k$$

are all globally generated. Note Λ_n is trivial. Then we can form

$$\Lambda_1^{a_1} \otimes \cdots \otimes \Lambda_{n-1}^{a_{n-1}}$$

for $a_i \geq 0$. These are exactly the globally generated ones, and the flexible ones as well. These are not all ample. The reason is, for example, that Λ_k is really the exterior power:

$$\Lambda_k = \wedge^k \left(F_n/F_{n-k}\right)$$

so it sort of only depends on F_{n-k}. We can project

$$G/B \to \mathrm{Gr}_{n-k}^n(\mathbb{C})$$

[1] This will be important for the proof of Serre's vanishing theorem.

by sending $F_* \mapsto F_{n-k}$. Now if we write $p^*V = F_n/F_{n-k}$, then $\wedge^k(V)$ is ample over $\mathrm{Gr}^n_{n-k}(\mathbb{C})$, but not when we pull it back.

Now if we define

$$L_\rho = \Lambda_1 \otimes \cdots \otimes \Lambda_{n-1} \ ,$$

then this is ample. In this picture the Picard group is somehow $\mathbb{Z}^{n-1}$, then the first quadrant gives us the globally generated bundles, and and the interior of this quadrant consists of the ample bundles.

Inside this B there is also the Cartan torus T consisting of diagonal matrices. Then characters of this are the weights. In fact $B = T \ltimes U$ (where U consists of strictly upper triangular matrices) since U is the kernel of $B \to T$. Then given a weight we have

$$B \to T \to \mathbb{C}^\times \ .$$

But a weight here is just a monomial $z_1^{a_1} \cdots z_n^{a_n}$ in the diagonal entries.

Given this situation we have the action $\mathbb{C}^\times \circlearrowright \mathbb{C}$ by scalar multiplication, and now we can form the orbit space

$$G \times_B \mathbb{C} = \left\{ (g,x) \in G \times \mathbb{C} \,|\, (gb,x) = \left(g, b^{-1}x\right) \right\}$$

which still maps to G/B. Then every fiber will be a copy of $\mathbb{C}$ in a non-canonical way.

So the story is, given a character $\lambda : T \to \mathbb{C}^\times$, there is action of B on $\mathbb{C}$, then $G \times_B \mathbb{C}_\lambda \to G/B$ gives a line bundle on G/B which are exactly the globally generated line bundles up to a sign.[2]

Example 2. Let C be a general curve in $\mathbb{P}^n$ isomorphic to $\mathbb{P}^1$. Note that some n will always exist since any curve can be embedded in $\mathbb{P}^3$, but this will work for any n.

If we take a point this is a closed subvariety, and we can consider :W the ideal sheaf. Locally there is just one coordinate x. So this is a principal ideal in an integral domain, and therefore it is isomorphic as a module to just $\mathcal{O}$. So the ideal sheaf is a line bundle $\mathcal{I} = \mathcal{L}^{\otimes -1}$. If we're on a curve and two points give rise to the same line bundle, then it must mean that there is a section of this line bundle that vanishes at one point and a pole on the other and vice versa. This is like a function to $\mathbb{P}^1$. This could be some sort of multiple covering, but as long as this isn't a double zero, it is locally 1-to-1, which turns out to mean it's an isomorphism.

This tells us that this line bundle is very much not globally generated. But it actually is ample, because our curve was embedded in $\mathbb{P}^n$. On this we have $\mathcal{O}(1)$ which is very ample, but sections of $\mathcal{O}(1)$ have zero locus given by a hyperplane. So if we just pick a hyperplane meeitng the curve in a finite set of distinct points, then this bunch of points must be ample.

The upshot is that the ideal sheaf of a single point won't even be globally generated, but if we take it of multiple points it will be ample. And if we take high powers it will be $\mathcal{O}(1)$ for some projective embedding.

On any curve which is not $\mathbb{P}^1$, there will be a line bundle whose only global section vanishes at a single point, but powers of this will give a projectie embedding, i.e. it is ample, even though it is far from beng globally generated.

[2]This is because a fundamental mistake was made in the very beginning of this subject: we really should have taken B to be lower triangular matrices!