LECTURE 23
MATH 256B

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We will continue talking about ample line bundles and define quasi-projective morphisms and varieties.

1. Quasi-projective varieties and morphisms

1.1. **Basic properties of (relative) ample line bundles.** So we have this concept of $\mathcal{L}$ being ample relative to a morphism $q : X \to T$ and it's somehow built into the definition that q should be quasi-compact.

Consider the following situation

$$\begin{array}{ccc} X' = T' \times_T X & \xrightarrow{\beta} & X \\ \downarrow{\scriptstyle q'} & & \downarrow{\scriptstyle q} \\ T' & \xrightarrow{\alpha} & T \end{array} .$$

If $\mathcal{L}$ is ample for q we want to consider whether or not there is some $\mathcal{L}'$ which is ample for q'. An initial guess might be $\mathcal{L}' := \beta^* \mathcal{L}$. As it turns out this works.

Similarly we can consider the following:

$$\begin{array}{ccc} X_1 \times_T X_2 & \longrightarrow & X_1 \\ \downarrow & \searrow^{q} & \downarrow{\scriptstyle q_1} \\ X_2 & \xrightarrow{q_2} & T \end{array}$$

where the X_i have line bundles $\mathcal{L}_i$ ample for q_i. The obvious guess is then that

$$p_1^* \mathcal{L}^{\otimes n} \otimes p_2^* \mathcal{L}_2^{\otimes n}$$

is ample on $X_1 \times_T X_2$ for q. As it turns out this works.

Now consider two quasi-compact maps between schemes

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

Then consider a line bundle $\mathcal{K}$ on Y ample for g and $\mathcal{L}$ on X ample for f. Then the question is if there is some line bundle on X ample for $g \circ f$. The obvious choice, $\mathcal{L} \otimes f^* \mathcal{K}$, doesn't actually work as we show in the following example.

Example 1. Consider the case

$$\mathbb{P}^1_k \times \mathbb{P}^1_k \xrightarrow{p_2} \mathbb{P}^1_k \xrightarrow{g} \operatorname{Spec} k$$

where $\mathcal{K} = \mathcal{O}(1)$. Write

$$\mathcal{O}(k, l) = p_1^* \mathcal{O}(k) \otimes p_2^* \mathcal{O}(l) \ .$$

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Then $\mathcal{L} = \mathcal{O}(1,0)$ is ample for f and in particular:

$$\mathcal{L} \otimes f^*\mathcal{K} = \mathcal{O}(1,1) \ .$$

But now we can take $\mathcal{L} = \mathcal{O}(1,-100)$ which still looks like $\mathcal{O}(1)$ on the fibers so is still ample, and then we have

$$\mathcal{L} \otimes f^*\mathcal{K} = \mathcal{O}(1,-99) \ .$$

which isn't ample for the map to the point.

Then the actual statement is the following:

Proposition 1. *If Z is quasi-compact, there exists an $n \gg 0$ such that*

$$\mathcal{L} \otimes f^*\mathcal{K}^{\otimes n}$$

on X is ample for $g \circ f$.

Proof. First we reduce to the case that Z is affine. If we cover Z with affines and know the affine case, then since Z is quasi-compact the max of the ns for each affine will be the n we want globally. Because it is ample (relative to g), $\mathcal{K}$ is flexible relative to g, i.e. for $U \subseteq Z$ affine, $\mathcal{K}$ is flexible on $g^{-1}U$, which implies $f^*\mathcal{K}$ is flexible on $(g \circ f)^{-1}(U)$. From this we see that if $\mathcal{L} \otimes f^*\mathcal{K}^{\otimes n}$ is ample then $\mathcal{L} \otimes f^*\mathcal{K}^{\otimes n+1}$ is also ample.

So we know we have enough sections $s \in \mathcal{K}^d$ such that Y_s are affine. If we write $s' = f^*s$ we have $X_{s'} = f^{-1}(Y_s)$, and in particular the relative ampleness tells us that we have enough $t \in \mathcal{L}^{\otimes e}(X_{s'})$ such that $X_{s't}$ are affine. We also have that $t \in f_*\mathcal{L}^{\otimes e}(Y_s)$ which is a quasi-coherent sheaf since f is quasicompact and separated. This means although this isn't a global section of this sheaf, if we multiply by a sufficiently high power m of s, we have that $t \otimes s^{\otimes m}$ extends to the global section $t' \in (f_*\mathcal{L}^{\otimes e}) \otimes \mathcal{K}^{\otimes md}(Y)$ and in particular we have a map:

$$\begin{array}{c} (f_*\mathcal{L}^{\otimes e}) \otimes \mathcal{K}^{\otimes md}(Y) \\ \downarrow \\ f_*\left(\mathcal{L}^{\otimes e} \otimes f^*\mathcal{K}^{\otimes md}\right) \end{array}$$

so $t' \mapsto t''$ which also extends our t. Now $\mathcal{L}^{\otimes d} \otimes f^*\mathcal{K}^{\otimes nd} = (\mathcal{L} \otimes f^*\mathcal{K}^{\otimes n})^{\otimes d}$ is ample so we are done. □

1.2. **Finiteness condition.** In addition to these basic properties of morphisms with ample bundles we also want some kind of finiteness. In particular, we want to consider $\mathcal{L}$ ample for $q : X \to T$ of finite type. This just means it is locally of finite type and quasi-compact. Recall locally of finite type means the following. So we can cover T with affines $\operatorname{Spec} A$ and their preimages with affines $\operatorname{Spec} B$. So we get maps $B \to A$, and then being locally of finite type means B must be a finitely generated A-algebra.

So the idea is that under the above conditions we can get:

$$\begin{array}{ccc} X & \overset{i}{\hookrightarrow} & Y = \operatorname{Proj} R \\ & {\scriptstyle q}\searrow \quad \swarrow & \\ & T = \operatorname{Spec} A & \end{array}$$

And as always, once we can do it for finitely many elements, we can make all of these ds the same, and for fixed s we can make all of the ns the same, but then we might as well make $n = 1$, so we can do it for all $s, t \in \mathcal{L}^{\otimes d}(X)$. So in fact we can make R look like $A[x_1, \cdots, x_m]/I$ so X is a locally closed subscheme of $\mathbb{P}^m_T$.

If T isn't affine it's more subtle. For T quasicompact we can say something, if T is both quasicompact and quasiseparated we can say even more, and if T is quasicompact and has itself an ample bundle, so it is separated, we can say something as well. We will see this in more detail next time.