LECTURE 24
MATH 256B

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1. Quasi-projective morphisms

Recall a morphism $q : X \to T$ is quasi-projective if it is of finite type,[1] and has an ample sheaf. We could just say locally of finite type because somehow part of the definition of an ample sheaf sort of requires it to be quasicompact anyway.

Recall last we saw that if we have $X \xrightarrow{f} Y \xrightarrow{g} Z$ and an ample sheaf on X for f, and an ample sheaf on Y for g, then as long as Z is quasicompact, there is an ample sheaf on X for $g \circ f$. As such f and g being quasi-projective implies $g \circ f$ is quasi-projective for quasicompact Z. We also saw that quasi-projective morphisms are closed under base extension and products.

1.1. Affine base.

Now let $T = \operatorname{Spec} A$. Then for q of finite type, we have

$$\begin{array}{ccc} X & \longrightarrow & Y = \operatorname{Proj} R \\ & \searrow \quad \swarrow & \\ & \operatorname{Spec} A & \end{array}$$

where R is a finitely generated graded A-algebra. In fact we can even have R_d generate R and then we get $R_d \to \Gamma\left(\mathcal{L}^{\otimes d}\right)$. Then for $\mathcal{E} = \tilde{R}_d$ we have $q^* \mathcal{E} \twoheadrightarrow \mathcal{L}^{\otimes d}$. But then we have that

$$q^* \tilde{R} = q^* \operatorname{Sym}(\mathcal{E}) / \mathcal{I}$$

for some ideal $\mathcal{I}$ which means $\operatorname{Proj} R \hookrightarrow \operatorname{Proj}(\operatorname{Sym} \mathcal{E}) = \mathbb{P}(\mathcal{E})$.

The idea is that for $\mathcal{E}$ locally free, taking Spec of the symmetric algebra gives the vector bundle with sections dual to this original bundle. So when we take Proj we're getting a sort of projective space bundle. In particular if $\mathcal{E}$ is locally finitely generated, then it is a closed subscheme of a '$\mathbb{P}^n$-bundle.'

$$\begin{array}{ccc} X & \longrightarrow & Y = \mathbb{P}(\mathcal{E}) \subseteq \mathbb{P}^n_A \\ & \searrow \quad \swarrow & \\ & \operatorname{Spec} A & \end{array}$$

The point is that when T is affine tensor powers of the ample bundle give us an embedding into projective space.

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[1] I.e. locally of finite type and quasicompact.

1.2. **Very ample line bundles.** We now make a definition which captures the good behaviour of the previous considerations.

Definition 1. We say that $\mathcal{L}$ is *very ample* for $q : X \to T$ if there exists $\mathcal{E} \in \mathbf{QCoh}(T)$ and the map induced[2] by $q^*\mathcal{E} \twoheadrightarrow \mathcal{L}$,

$$\begin{array}{ccc} X & \longrightarrow & \mathbb{P}(\mathcal{E}) \\ & {\scriptstyle q}\searrow \quad \swarrow & \\ & T & \end{array}$$

is an embedding (locally closed).

Note that assuming q is quasicompact, very ample implies ample. If $T = \operatorname{Spec} A$ and q is quasi-projective, then $\mathcal{L}$ is ample iff some $\mathcal{L}^{\otimes d}$ is very ample. So now we want to see if T isn't affine.

1.3. **Non-affine base.** So now consider quasicompact $q : X \to T$ with some ample bundle $\mathcal{L}$. Then cover T with open sets, and assume $\mathcal{L}$ is very ample on all of these open sets. Then we claim that it is very ample in general. The fact that there is an ample bundle at all means q is separated so $q_*\mathcal{L}$ is qcoh. Having a map $q^*\mathcal{E} \to \mathcal{L}$ corresponds to a map $\mathcal{E} \to q_*\mathcal{L}$, so we have

$$q^*\mathcal{E} \to q^*q_*\mathcal{L} \twoheadrightarrow \mathcal{L} \ .$$

This gives us

$$X \to \mathbb{P}(q_*\mathcal{L}) \to \mathbb{P}(\mathcal{E})$$

and we know that this composition is a locally closed embedding, which implies the first map is a locally closed embedding.

The upshot of this is that if there is any $\mathcal{E}$ that works, then $q_*\mathcal{L}$ works.

So now we could restate the definition (in the case that q is quasicompact) that $\mathcal{L}$ is very ample if the canonical morphism $q^*q_*\mathcal{L} \twoheadrightarrow \mathcal{L}$ is an embedding.

Now we have that being ample is the same as some power is very ample, but this isn't quite local because we might have different powers in different local patches. But if T is quasicompact we can take some large common power. Formally:

Fact 1. *If $q : X \to T$ is quasi-projective and T is quasi-compact, then $\mathcal{L}$ is ample iff some $\mathcal{L}^{\otimes d}$ is very ample.*

1.4. **Finite generation.** In the affine case we were able to actually make this correspond to a finitely generated module (since this R was a f.g. algebra). So let's see what we can do here. So we know we can do it all for one $\mathcal{E}$, namely $q_*\mathcal{L}$, and we know that each of these are locally finitely generated, but they might not be locally generated in the same way globally. As it turns out, as long as T is quasicompact, we can take the direct limit of the locally f.g. subsheaves. But this won't give us the actual qco sheaf for a general quasicompact scheme. We need it to be quasiseparated as well.

Proposition 1. *If T is quasicompact and quasiseparated then every $\mathcal{E} \in \mathbf{QCoh}(T)$ is $\varinjlim$ of its locally finitely generated subsheaves.*

[2] Such a map is always induced because $q^*\mathcal{E} \twoheadrightarrow \mathcal{L}$ corresponds to $q^* \operatorname{Sym} \mathcal{E} \twoheadrightarrow \mathcal{O}_X[\mathcal{L}]$ and $\mathbb{P}(\mathcal{E}) = \underline{\operatorname{Proj}} \operatorname{Sym}(\mathcal{E})$.

Proof. We want to show that every section of $\mathcal{E}$ on every open set, $s \in \mathcal{E}(U)$, is a section of one of these subsheaves. Of course there is no problem on U because we can just take

$$\mathcal{O}_U \longrightarrow \mathcal{E}|_U$$

$$1 \longmapsto s$$

but the image of this is locally finitely generated subscheme of $\mathcal{E}|_U$ not $\mathcal{E}$. So now we want to extend this. Since T is quasicompact, we can write

$$T = \bigcup_{i=1}^{n} U_i \ .$$

Then assume U is of the form

$$U = U_1 \cup \cdots \cup U_k \ .$$

Then for $\mathcal{E}'' \subseteq \mathcal{E}|_U$ it is enough to extend $\mathcal{E}''$ to some $\mathcal{E}' \subset \mathcal{E}|_{U \cup U_{k+1}}$.

So now the problem is that we have an affine $V = \operatorname{Spec} A$, $W \subseteq V$ a quasicompact open subset, and then we have $\mathcal{E} = \tilde{E}$ for some module E (since this is an affine scheme). Then we have $\mathcal{E}'' \subseteq \mathcal{E}|_W$. Now we want a submodule of E locally generated such that $\tilde{E}$ agrees with $\mathcal{E}''$. We know

$$j_*\mathcal{E}'' \subseteq j_*j^*\mathcal{E} = j_* \ \mathcal{E}|_W$$

are both quasicompact, and that we have a canonical map $\varphi\mathcal{E} \to j_*j^*\mathcal{E}$ (since j_* and j^* are adjoint) which takes a section and just restricts it. Then we have $j_*\mathcal{E}'' \subseteq j_*j^*\mathcal{E}$ and we can take $\mathcal{F} = \varphi^{-1}(j_*\mathcal{E}'') \subseteq \mathcal{E}$ and then $\mathcal{F} = \tilde{F}$ for some module and in particular $\tilde{F}\Big|_W = \mathcal{E}''$ so it is the right sheaf, but it might not be finitely generated. Since W is quasicompact we can write

$$W = \bigcup_{i=1}^{l} V_{f_i} \ .$$

Then $\tilde{F}_{f_i} = \mathcal{E}''|_{V_{f_i}}$ is locally finitely generated, so F_{f_i} is a f.g. A_{f_i}-module generated by $g_{ij}/f_i^{n_j}$. Now take $E' \subseteq F$ generated by the g_{ij}. Notice that we still have

$$\tilde{E}'\Big|_W = \mathcal{E}''$$

so E' gives us this whole subsheaf and is finitely generated. $\square$