LECTURE 27
MATH 256B

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1. Last time

Recall we observed that if we have a quasicompact morphism $X \xrightarrow{p} Y$ then there is a local ring criterion for whether a point is in the image. This was that $y \notin \overline{p(x)}$ iff its infinitesimal local neighborhood has empty preimage, in the sense that

$$(\operatorname{Spec} \mathcal{O}_{X,y}) \times_Y X = \emptyset \ .$$

We also observed that for finite morphisms Nakayama's lemma implied that these have some sort of 'Nakayama property' which was that

$$(\operatorname{Spec} \mathcal{O}_{X,y}) \times_Y X = \emptyset \iff p^{-1}(y) = \emptyset \ .$$

Putting these together shows us that a point is not in the closure iff it is not in the image. I.e.

$$p(x) = \overline{p(x)} \ .$$

We can enhance this slightly by observing that if we have a closed subscheme $Z \subseteq X$ then the composition $Z \subseteq X \xrightarrow{p} Y$ is also finite, which means p is a closed map since it sends closed sets to closed sets.

2. Nakayama's lemma and elimination theory

Now consider $X = \operatorname{Proj} R \xrightarrow{p} Y$. Let R be a qco graded $\mathcal{O}_Y$-algebra, locally f.g., with R_0 locally a f.g. $\mathcal{O}_Y$-module. Note these conditions imply that in fact every R_d is a f.g. $\mathcal{O}_Y$-module. The typical situation is that $R_0 = \mathcal{O}_Y$.

Now what can we say about this morphism? First of all this will be a quasicompact morphisms because of this local finite generation. In fact p is of finite type (which implies quasicompact). Then we also claim that this has the Nakayama property. This comes down to the fact that these are finitely generated modules, so Nakayama applied to them.

So let $(A, \mathfrak{m}) = \mathcal{O}_{Y,y}$ be a local ring, $k = A/\mathfrak{m}$, then let R be a graded A-algebra. So assume $\operatorname{Proj}(k \otimes_A R) = \emptyset$ since this is the preimage $p^{-1}(y)$. Write $S = k \otimes_A R$. Then we want to show $\operatorname{Proj} S$ is empty. This means $V(S_+) = \operatorname{Spec} S$. But this means that S_+ is generated over S_0 by finitely many nilpotent elements. This implies that there exists some N such that $S_d = 0$ for all $d > N$. In this case $S_d = k \otimes_A R_d$, and R_d was a finitely generated module over the local ring A, so by Nakayama's lemma this implies $R_d = 0$ for $d > N$. Therefore the image of p is closed. Then as before we have the following:

Theorem 1. *If $X = \underline{\operatorname{Proj}}(R) \xrightarrow{p} Y$ with R as above, then p is a closed map.*

This by itself gives us elimination theory. Classically the picture is somehow

$$\begin{array}{c} Z \subseteq \mathbb{P}^n_k \times_k \mathbb{A}^m_k \\ \downarrow \\ \mathbb{A}^m_k \end{array}$$

for Z closed:

$$Z = \operatorname{Proj} k\left[a_1, \dots, a_m, x_0, \dots, x_n\right]/J \to \operatorname{Spec} k\left[a_1, \dots, a_n\right] \ .$$

Then the conclusion is that the image of this map is closed. So this theorem implies classical elimination theory, and in some sense is equivalent to it.

Example 1. A finite morphism $X = \operatorname{Spec} A \xrightarrow{p} Y$ is a special case of this since $X = \operatorname{Proj}\left(A\left[t\right]\right)$.

If we base extend p:

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "p'"] & X = \operatorname{Proj} k \arrow[d, "p"] \\ Y' \arrow[r, "q"] & Y \end{tikzcd}$$

we get that p' is also closed since $X = \operatorname{Proj} k'$ where $k' = q^* k$. As in this case, if a morphism has a property and every base extension has the property we say that the morphism universally has the property. Note that closed maps are not trivially universal (as other features or morphisms are) so there is content to this statement.

In this situation there are a couple other things we know about p. It is of finite type (quasicompact and locally of finite type). This is automatically a universal thing. It is also separated, since any Proj to a base is a separated morphism. Again base extensions of separated morphisms are always separated, so it is automatically universal. Therefore we haver that p is universally closed, of finite type, and separated, so it is in fact proper. This turns out to sort of be the right thing to consider.

3. Yoga of morphisms

We will somehow use the previous discussion as a prototype, but have a general discussion about the yoga of morphisms. Consider the class of universally closed morphisms. Of course this is closed under base extension by definition. It is also closed under composition, and includes the closed embeddings.

Now we do the yoga. Note that this all holds for any any class of morphisms which has these three properties. Examples include affine, quasicompact, finite-type, separated, proper (since it applied to the previous three) and many more morphisms. We will see that given a morphism of two schemes over a given base:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"] & & Y \arrow[dl, "q"] \\ & Z & \end{tikzcd}$$

such that p is universally closed, and q is separated, then this implies f is universally closed.

Consider the graph of f. We can think of this as a map

$$\Gamma_f : X \to X \times_Z Y \ .$$

Then we can project to Y:

$$\begin{tikzcd} X \arrow[r, "\Gamma_f"] \arrow[rr, bend right, "f"'] & X \times_Z Y \arrow[r, "\pi"] & Y \end{tikzcd} \quad . \tag{1}$$

One way to think of π is as a base extension of f itself:

$$\begin{tikzcd} X \times_Z Y \arrow[r] \arrow[d, "\pi"] & X \arrow[d, "p"] \\ Y \arrow[r, "q"] & Z \end{tikzcd} \quad .$$

Since p was universally closed, π is closed, and in fact universally closed.

As it turns out the map Γ_f is a base extension as well:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\Gamma_f"] & Y \arrow[d, "\Delta"] \\ X \times_Z Y \arrow[r, "(f,1)"] & Y \times_Z Y \end{tikzcd} \quad .$$

One can easily check this is a set theoretic fiber product. Then it is the scheme theoretic fiber product because the functor represented by this is the fiber product of the functors represented by the other schemes. Now the map $Y \to Z$ is separated, which means Δ is a closed embedding (it is always locally closed). Therefore Γ_f is universally closed. But since universally closed morphisms are closed under composition, from (1) we have that f is universally closed.

Now we can answer another motivating question which was the following. Consider $q : X \to T$ quasi-projective. Then we can take

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[dr, "q"'] & Y = \mathrm{Proj}\,(k) \arrow[d] \\ & T \end{tikzcd} \quad ,$$

and the question is when i is a closed embedding. We know it is locally closed, so we just need it to be a closed map. And in particular, i is a closed embedding precisely when q is universally closed. So now we understand which quasi-projective morphisms give a closed embedding, and in particular they just have to be proper.