LECTURE 28
MATH 256B

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1. Projective morphisms

Recall from last time that we call a morphism $q : X \to Y$ *projective* if it is quasi-projective and universally closed (i.e. proper). The point is that if we use the fact that it is quasi-projective to get

$$\begin{array}{ccc} X & \overset{i}{\hookrightarrow} & \operatorname{Proj}(R) \\ & {\scriptstyle q}\searrow \quad \swarrow & \\ & T & \end{array}$$

then i is a closed embedding. If T is affine, then we can make this R be a finitely generated algebra over whatever ring T was Spec of. In fact the morphism is proper iff the associated embedding is closed.

If we make assumptions about the base, things get progressively nicer. For T quasi-compact then we can get R to be generated in one degree:

$$\begin{array}{ccc} X & \overset{i}{\hookrightarrow} & \mathbb{P}(\mathcal{E}) \\ & {\scriptstyle q}\searrow \quad \swarrow & \\ & T & \end{array}$$

and if q is also quasi-separated, we can make $\mathcal{E}$ locally finitely generated. If we further assume that T has an ample sheaf (e.g. it is affine) then we can get it such that $\mathcal{E}$ direct sum of copies of $\mathcal{O}$, and we have $X \overset{i}{\hookrightarrow} \mathbb{P}^N_T \to T$.

Projective morphisms are almost composable. If we have $X \xrightarrow{f} Y \xrightarrow{g} Z$, then proper morphisms are closed under composition, so as long as Z is quasi-compact, then the composition is projective.

In addition, if $g \circ f$ is projective, and g is separated, then this implies f is projective.

Basically the story is the following. Consider the category **P** of projective schemes X over T where T is somehow nice such as quasi-compact and quasi-separated (or even just affine) then this is a nice category. In particular all morphisms are projective, and if T is affine, then they all have an ample sheaf.

2. Grassmannians

So now we're back in the world of $k = \overline{k}$. Then a Grassmann variety $G(n, r)$ is supposed to parameterize r-dimensional subspaces $V \subseteq k^n$.

Example 1. For $r = 1$, $G(n, 1) = \mathbb{P}^{n-1}$.

Note that there is a natural bijection between such $V \subseteq k^n$ and $V^\perp \subseteq (k^n)^*$, so we have a natural bijection

$$G(n,r) \cong G(n,n-r) \ .$$

So in the case of projective space, we can also think of this as $G(n,n-1) = \mathbb{P}^{n-1}$. In the case of $G(n,1)$ we get the tautological line bundle to be $\mathcal{O}(-1)$, and in the case of $G(n,n-1)$ we get $\mathcal{O}(1)$ which is somehow better because it has more sections.

2.1. **Structure as a variety.** So we want to somehow put coordinates on this. For any $V \subseteq k^n$ of dimension r choose a basis $v_1,\dots,v_r$ of V. Then we can form a matrix with r rows and n columns:

$$M = \begin{pmatrix} v_1 \\ \cdots \\ v_r \end{pmatrix} \ .$$

Then we can consider $I \subseteq [n]$ where $|I| = r$. Write $\delta_I = \det M_I$. Note these are not all 0. But there are choices being made here. I.e. we might have another M' for the same V. But the new vectors are linear combinations of the old ones, so we can write down a change of basis matrix, i.e. $M' = gM$, and in particular $g \in \mathrm{GL}_r$. So we get new coordinates δ'_I which are related to the old ones by:

$$\delta'_I = \det(M'_I) = \det(gM_I) = \det(g)\,\delta_I \ .$$

So if we write down these coordinates, we get a point:

$$(\cdots : \delta_I : \cdots) \in \mathbb{P}^N$$

for $N = \binom{n}{r} - 1$ which only depends on V as a point of projective space. These coordinates are called the *Plücker coordinates*. Note this is well-defined since they cannot all be 0. So now we have a map

$$G(n,r) \to \left(\mathbb{P}^N_k\right)_{\mathrm{cl}} \ .$$

So now we want to figure out if this is injective, and if it is defined by homogeneous equations. The answer is yes and yes, but first we will work out an explicit example.

2.2. **Explicit example.** We will calculate $G(4,2)$. This is the first example which isn't just projective space. This will be a proper subvariety of projective space. In this case our matrices look like

$$M = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{pmatrix}$$

and the δ_{ij} are

$$\delta_{ij} = \det \begin{pmatrix} x_i & x_j \\ y_i & y_j \end{pmatrix} \ .$$

We will have six of them. If $\delta_{12} \neq 0$ WLOG

$$M = \begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{pmatrix}$$

and then

$$\delta_{12} = 1 \qquad \delta_{23} = -a \qquad \delta_{13} = c$$

$$\delta_{24} = -b \qquad \delta_{14} = d \qquad \delta_{34} = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

which are related by the following polynomial equation:

$$\delta_{12}\delta_{34} + \delta_{14}\delta_{23} - \delta_{13}\delta_{24} = 0 \ .$$

Such a relation is called a *Plücker relation*. The point is $X \subseteq \mathbb{P}^5$ is defined by

$$X = V\left(\delta_{12}\delta_{34} + \delta_{14}\delta_{23} - \delta_{13}\delta_{24}\right)$$

and then

$$\mathrm{Gr}\,(4,2) \longrightarrow X \hookrightarrow \mathbb{P}^5 \ , \qquad \mathrm{Gr}\,(4,2) \xrightarrow{\ j\ } \mathbb{P}^5 \ .$$

We can write

$$j^{-1}\left(U_{\delta_{12}}\right) \xrightarrow{\simeq} X_{\delta_{12}} \cong \mathbb{A}^4 = \mathrm{Spec}\, k\,[a,b,c,d]$$

and by symmetry this works for the other opens. So $\mathrm{Gr}\,(4,3)$ is somehow a natural projective variety.

Without going through the details, we discuss doing this with large matrices. Consider $\mathrm{Gr}\,(7,4)$. This has $\binom{7}{4}$ Plücker coordinates. Then for any matrix we can WLOG write it as:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & a & b & c \\ 0 & 1 & 0 & 0 & d & e & f \\ 0 & 0 & 1 & 0 & g & h & i \\ 0 & 0 & 0 & 1 & j & k & l \end{pmatrix} \ .$$

Now we can play the same game. Any minor will have k columns from the identity matrix, and l interesting columns. Then the determinant will just be the same as the determinant of some $l \times l$ minor of the interesting part of the original matrix. Then the story holds effectively the same way.

Basically what happens is we take our favorite two rows and favorite two columns. Then we multiple it by a choice of $r-2$ rows and columns of the identity matrix to get a homogeneous quadratic. So, for example, in this case we get

$$\delta_{1234}\delta_{1457} = \pm\delta_{1345}\delta_{1247} \pm \delta_{1347}\delta_{1245} \ .$$

So the quadratics always somehow look like three terms.

The general story covers $\mathrm{Gr}\,(7,4)$ with affines $\mathbb{A}^{r(n-r)}$. We haven't defined smoothness and dimension, but whatever they mean certainly they should mean that this thing is smooth and of dimension $r\,(n-r)$.