LECTURE 29
MATH 256B

LECTURE: PROFESSOR MARK HAIMAN
NOTES: JACKSON VAN DYKE

Last time we saw an ad hoc way to see the Grassmannian as a projective variety. Today we will do this in a more general way. In particular we will define a functor and show there is a scheme which represents it.

1. Grassmannians

Consider some scheme S and a quasi-coherent sheaf ($\mathcal{O}_S$-module) $\mathcal{M}$.

Example 1. The motivating example is just $S = \operatorname{Spec} k$ $\mathcal{M} = \widetilde{k^n}$.

We now define the following functor

$$\underline{\underline{G(\mathcal{M}, r)}} : (\mathbf{Sch/S})^{\mathrm{op}} \to \mathbf{Set}$$

as follows. For $\begin{array}{c} X \\ \downarrow p \\ S \end{array}$ we assign the set of (qco)[1] subsheaves $\mathcal{N} \subseteq p^* \mathcal{M}$ such that $p^* \mathcal{M} / \mathcal{N}$ is locally free of frank r i.e. locally it is just $\mathcal{O}_X^r$.

Now we see this is actually a functor. So consider

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr, "q"'] & & X \arrow[dl, "p"] \\ & S & \end{tikzcd} .$$

Then we have $q^* \mathcal{M} = f^* p^* \mathcal{M}$ and we can apply f^* to the following exact sequence:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{N} \arrow[r] & p^* \mathcal{M} \arrow[r] \arrow[d, squiggly, "f^*"] & \mathcal{V} \arrow[r] & 0 \\ 0 \arrow[r] & f^* \mathcal{N} \arrow[r] & q^* \mathcal{M} \arrow[r] & f^* \mathcal{V} \arrow[r] & 0 \end{tikzcd} .$$

We get exactness for free at $f^* \mathcal{N}$, but it is actually exact elsewhere. This can be seen by a simple Tor argument. It can also be seen directly. Let $y \in Y$ (and $x = f(y)$). First decompose the middle element of the first sequence

$$p^* \mathcal{M}_x = \mathcal{N}_x \oplus \mathcal{V}_x \ .$$

Then we have that this functor is

$$\left(\mathcal{O}_{Y,y} \otimes_{\mathcal{O}_{X,x}} - \right)$$

which preserves direct sums, so we have that the middle term of the second sequence also splits.

[1] This is in parentheses because because $p^* \mathcal{M}$ is qco anyway, and if we have an exact sequence where the quotient is qco, then the kernel is as well. So we don't need to separately insist on this.

Now we claim $\underline{\underline{G(\mathcal{M},r)}}$ is a sheaf in the Zariski topology. This is almost obvious. First cover $X = \cup_\alpha X_\alpha$. We have some $\mathcal{N} \subseteq p^*\mathcal{M}|_{X_\alpha}$ for each α such that

$$\mathcal{N}_\alpha|_{X_\alpha \cap X_\beta} = \mathcal{N}_\beta|_{X_\alpha \cap X_\beta} \ .$$

So we have a global $\mathcal{N}$ such that

$$\mathcal{N}_\alpha = \mathcal{N}|_{X_\alpha} \ .$$

Now we want to somehow make S affine. Consider some open $W \subseteq S$. This gives a subfunctor

$$\underline{\underline{G_W}} \subset \underline{\underline{G(\mathcal{M},r)}}$$

defined to be:

$$\underline{\underline{G_W}}(X) = \begin{cases} \emptyset & p(x) \not\subseteq W \\ \underline{\underline{G(\mathcal{M},r)}}(X) & p(X) \subseteq W \end{cases} .$$

We claim $\underline{\underline{G_W}}$ is in fact an open subfunctor. Recall this means the following. Let X over S and consider a functorial map[2] $\underline{\underline{X}} \to \underline{\underline{G(\mathcal{M},r)}}$. For

$$\begin{array}{ccc} T & \xrightarrow{\varphi} & X \\ & \searrow^{q} & \downarrow p \\ & & S \end{array}$$

we have

$$\underline{\underline{X}}(T) \to \underline{\underline{G(\mathcal{M},r)}}(T) \supseteq \underline{\underline{G_W}}(T)$$

and for

$$U = p^{-1}(W)$$

we have

$$\underline{\underline{U}}(T) = \underline{\underline{G_W}}(T) \ .$$

The point is that we can assume S is affine.

Consider

$$R^{(J)} \to R^{(I)} \to M \to 0 \ .$$

Let ξ_j give the defining relations on the generators x_i of M:

$$M = R \cdot \{x_i \,|\, x_I\} / (\xi_j) \ .$$

Let $\underline{\underline{G_B}} \subseteq \underline{\underline{G\left(\tilde{M},r\right)}}$ be the following subfunctor. Let $B \subseteq I$ such that $|B| = r$. For a scheme X over S, map this to $\mathcal{N} \subseteq p^*\tilde{M}$ such that $p^*\tilde{M}/\mathcal{N}$ is free on $\{x_b \,|\, b \in B\}$. The claim is that $\underline{\underline{G_B}}$ is an open subfunctor represented by an affine scheme over S. We want to look at any $p : X \to S$. First consider $X = \operatorname{Spec} A$ affine: $\underline{\underline{G_B}}(\operatorname{Spec} A)$. So consider $p : X \to S = \operatorname{Spec} R$. This corresponds to $R \to A$. Then

$$p^*\tilde{M} = \widetilde{A \otimes_R M}$$

somehow contains the $1 \otimes x_i \in A \otimes_R M$ which we just write as x_i and

$$\xi_i \mapsto \sum_i r_{ij} x_i \ .$$

[2]Recall that by Yoneda such a map is the same as an element $\mathcal{N} \in \underline{\underline{G(\mathcal{M},r)}}(X)$.

So now we want a quotient sheaf which corresponds to a quotient of this module:

$$0 \to N \to A \otimes M \to V \to 0$$

such that $V \cong A^r$ such that x_b for $b \in B$ map to the unit vectors of A^r. Certainly we have

$$x_i \mapsto \sum a_{ib} x_b \pmod N$$

for some $a_{ib} \in A$. But these are not arbitrary. We have the following conditions:

(1) $a_{ib} = \delta_{ib}$ for $i \in B$.
(2) $\sum r_{ji} a_{ib} = 0$ for all $j \in J$ and $b \in B$.

Certainly we can map

$$A^{(I)} \to A^r \ ,$$

and the conditions say exactly that it factors through the map $A^{(I)} \to A \otimes M$, and we get that

$$N = \left(x_i - \sum a_{ib} x_b\right) \ .$$

So we get a one-to-one correspondence between the set $\underline{\underline{\underline{G_B}}}(\operatorname{Spec} A)$ and solutions to these linear equations. I.e. for

$$H = R\left[a_{ib} \mid i \in I, b \in B\right] / \left(a_{ib} - \delta_{ib}, i \in B \mid \sum r_{ji} a_{ib}, j \in J, b \in B\right)$$

we have

$$\underline{\underline{\underline{G_B}}}(\operatorname{Spec} A) = \operatorname{Hom}_{R\text{-}\mathbf{Alg}}(H, A) = \underline{\underline{\underline{\operatorname{Spec} H}}}(\operatorname{Spec} A) \ .$$

Now we want to consider

$$\underline{\underline{\underline{\operatorname{Spec} H}}}(X)$$

for

$$\begin{tikzcd} X \arrow[r, "p"] \arrow[d] & S = \operatorname{Spec} R \\ \operatorname{Spec} H \arrow[ur] & \end{tikzcd} \ .$$

This is in correspondence with

$$H \to \mathcal{O}_X(X) \ .$$

Now cover $X = \cup X_\alpha$. Then we have R-algebra homomorphisms $H \to A_\alpha$. So for ever α we have an element of $\underline{\underline{\underline{G_B}}}(X_\alpha)$, the collection of which are compatible on intersections. This means we have a one-to-one correspondence between $\underline{\underline{\underline{G_B}}}(X)$ and $\underline{\underline{\underline{\operatorname{Spec} H}}}(X)$.

We have some more work to do, but to close today, consider the case that $\mathcal{M}$ is free

$$\mathcal{M} = \mathcal{O}_S^n \ .$$

Then $p^*\mathcal{M} = \mathcal{O}_X^n$ for $p : X \to S$. Then in this case the first type of relation is encapsulated in the matrix

$$\begin{pmatrix} \mathrm{id}_{r\times r} & a_{ib} \end{pmatrix}$$

and the second type don't exist here. So $\operatorname{Spec} H$ is just an affine space with these coordinates, and this looks quite a lot like the classical picture...

To be continued...