LECTURE 3
MATH 256B

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Today we will discuss some more examples and features of the Proj construction.

1. Examples of Proj

Example 1. As we saw last time, $\operatorname{Proj} k[x_0, \cdots, x_n]$ is classical projective space $\mathbb{P}^n_k$. This is covered by affines X_{x_i}, i.e. the same ad hoc affines we found to show it was a scheme to begin with. Explicitly these now look like:

$$X_{x_i} = \operatorname{Spec} k[x_0/x_i, \cdots, x_n/x_i] \cong \mathbb{A}^n_k$$

where we omit x_i/x_i. Note that $X_{x_i} \cap X_{x_j} = X_{x_i x_j}$. Note that classically we only saw this for $k = \overline{k}$, but this construction works more generally.

Example 2. For any k-algebra A, we can consider $Y = \operatorname{Spec} A$, and $\operatorname{Proj} A[x_0, \cdots, x_n]$. This will be covered by n-dimensional affines over A in exactly the same way:

$$X_{x_i} = \mathbb{A}^n_A = \operatorname{Spec} A[v_1, \cdots, v_n] \ .$$

But we can also think of this as

$$A[\underline{x}] = A \otimes_k k[\underline{v}]$$

so

$$\mathbb{A}^n_A = \operatorname{Spec} A \times \mathbb{A}^n_k$$

and therefore

$$\operatorname{Proj} A[x_0, \cdots, x_n] = Y \times \mathbb{P}^n_k \ .$$

In general, the Proj of something with nontrivial degree 0 part will be the product of Spec of this thing with projective space.

2. Functoriality

Recall that since we have the quotient map $R \to R/I$ we get that $\operatorname{Spec} R/I \hookrightarrow \operatorname{Spec} R$ is a closed subscheme. In light of this, we might wonder whether or not $\operatorname{Proj}(R/I) \hookrightarrow \operatorname{Proj}(R)$ is a closed subscheme. In fact it's not even quite clear if there is an induced map at all.

For R a graded ring and I a graded ideal, then the map $R \to R/I$ is naturally a grade ring homomorphism. More generally, for a graded ring homomorphism $B \to A$, we want to know if this gives us a map $X = \operatorname{Proj} A \to \operatorname{Proj} B = Y$. If we take some $f \in B_d$ for $d > 0$, then the open set $Y_f \subset Y$ corresponding to f is $Y_f = \operatorname{Spec}\left(B\left[f^{-1}\right]\right)_0$. Now for $\varphi : B \to A$, we get an induced map $B\left[f^{-1}\right] \to A\left[\varphi(f)^{-1}\right]$ which is still a graded homomorphism. In particular, it sends the degree 0 part to the degree 0 part, so we get a map $X_{\varphi(f)} \to Y_f$. That

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is, for each of the standard open sets in Y we found an open set in X which maps naturally to it. By construction this is sort of compatible on the overlaps.

The potential issue here is that the $X_{\varphi(f)}$ might not cover X. Recall that by definition:

$$X = \operatorname{Proj} A \hookrightarrow \operatorname{Spec}(A) \setminus V(A_+) \qquad Y = \operatorname{Proj} B \hookrightarrow \operatorname{Spec}(B) \setminus V(B_+) \ .$$

We know that we always have $\varphi(B_+) \subset A_+$ since it is a graded homomorphism, which implies $V(A_+) \subseteq V(\varphi(B_+))$. We also know that $\alpha : \operatorname{Spec} A \to \operatorname{Spec} B$ satisfies

$$V(\varphi(B_+)) = \alpha^{-1} V(B_+)$$

so we get:

$$V(A_+) \subseteq \alpha^{-1} V(B_+) \qquad\qquad \alpha(V(A_+)) \subseteq V(B_+) \ .$$

But if these are not equal, then $\alpha(\operatorname{Spec} A \setminus V(A_+))$ doesn't have to be contained in $\operatorname{Spec} B \setminus V(B_+)$. What we do have is that we can intersect $\operatorname{Spec}(A) \setminus \alpha^{-1}(V(B_+)) = \operatorname{Spec}(A) \setminus V(\varphi(B_+))$ with $\operatorname{Spec} A \setminus V(A_+)$ to get this sort of canonical open subset of X:

$$U = X \setminus V(\varphi(B_+)) \ .$$

Example 3. The geometry of this is as follows. Consider the graded ring homomorphism $k[x] \hookrightarrow k[x,y]$ Then the corresponding map from $\mathbb{P}^2 \to \mathbb{P}^1$ maps lines to lines, and the vertical line to a point. This is sort of silly, but in higher dimensions, the only maps from higher projective space to the projective line are constant maps.

Note that if $\sqrt{\varphi(B_+)} = A_+$ (e.g. $R \twoheadrightarrow R/I$) we have that $U = X \setminus V(A_+)$ so in this case, things work out fine. In particular, if

$$\left(R\left[f^{-1}\right]\right)_0 = \left((R/I)\left[f^{-1}\right]\right)_0$$

since we also have

$$(R/I)\left[f^{-1}\right] = R\left[f^{-1}\right]/I \ ,$$

then the kernel of this map is $\left(IR\left[f^{-1}\right]\right)_0$, and this is all compatible. Now the local picture on open subsets is that of an inclusion of a closed subscheme.

This means that any graded ideal $I \subseteq k[x_0, \cdots, x_n]$ will correspond to a closed subvariety $X \subseteq \mathbb{P}^n_k$. This is exactly the classical subscheme defined by some collection of homogeneous polynomials that we are expecting.

More generally, if we have some graded ring R, and $f \in R_1$, then $X_f = \operatorname{Spec} R\left[f^{-1}\right]_0$. Recall we studied this in two steps. First we took

$$R\left[f^{-1}\right]_{\mathbb{Z}d} \simeq R\left[f^{-1}\right]_0 \left[x^{\pm 1}\right] \ .$$

Then for we passed to $R\left[f^{-1}\right]_{\mathbb{Z}d}/(f-1)$ and for $d = 1$, $R\left[f^{-1}\right]/(f-1) = R/(f-1)$. Therefore as long as $f \in R_1$, we have $X_f = \operatorname{Spec} R/(f-1)$.

3. Weighted projective space

In Proj, two different ideals can define the same variety. This is immediate in the sense that if $\sqrt{I} \supseteq R_+$, then $V(I)$ is empty. In particular, (1) and $(x_0, \cdots, x_n)$ both have $V(I) = \emptyset$. More interestingly, we could also take $\left(x_0^2, \cdots, x_n^2\right)$ which also has empty vanishing locus. This is something to do with a large degrees phenomenon: if two ideals agree in large degree, they will define the same subscheme.

The following is a basic fact about projective space. Suppose we want to compare $\operatorname{Proj} R$ to the thinned out version $\operatorname{Proj} R_{\mathbb{Z}d}$ for $d > 0$. First of all $R_{\mathbb{Z}d} \subseteq R$, but we know this doesn't necessarily induce a map of Proj of these rings. However in this case, since $\varphi\left((R_{\mathbb{Z}d})_+\right) = (R_{\mathbb{Z}d})_+$, we know that $\sqrt{\varphi\left((R_{\mathbb{Z}d})_+\right)} = R_+$ so we do in fact get a map

$$X = \operatorname{Proj} R \to \operatorname{Proj} R_{\mathbb{Z}d} = Y\ .$$

Now notice that we always have that $V(f) = V\left(f^d\right)$, so $X_f = X_{f^d}$ and f^d is a homogeneous multiple of d. So X_f is covered by open sets of this form. In particular, $X_g \to Y_g$, and in local terms we have

$$(R_{\mathbb{Z}d})\left[g^{-1}\right]_0 \to R\left[g^{-1}\right]_0\ .$$

But these rings are the same, so the map $X_g \to Y_g$ is an isomorphism on pieces of the cover, so $\operatorname{Proj} R \xrightarrow{\sim} \operatorname{Proj} R_{\mathbb{Z}d}$ is an isomorphism.

This means that if R is some quotient, then this thinning only depends on the thinning of the ideal. I.e. $V(I)$ only depends on $I_{\mathbb{Z}d}$ for any $d > 0$.

To be continued...