LECTURE 30
MATH 256B

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1. Grassmann varieties

1.1. **Last time.** Consider a scheme S. Then we are considering some $\mathcal{M} \in \mathbf{QCoh}(S)$, and defined $\underline{\underline{G(\mathcal{M}, r)}}$ to take some scheme $q : X \to S$ over S and sends it to the set of locally free rank r quotients $q^*\mathcal{M}/\mathcal{N}$. Last time we explained how this was a functor (which used the local freeness) and we also discussed some parts of how we will represent this functor as a scheme. To do this we need to check it is a sheaf in the Zariski topology, which is almost obvious, and then we need to find some open subfunctors which cover it which are representable.

First we saw that if we took an open subset of S, then we could define an open subfunctor which is somehow its preimage, i.e. it's trivial away from the preimage. Specifically, for representability, we can assume $S = \operatorname{Spec} R$ and $\mathcal{M} = \tilde{M}$. Then we can let x_i generate M for $i \in I$, and let $\xi_i = \sum r_{ij} x_i$ be the defining relations.

Now for any $B \subseteq I$ such that $|B| = r$ we can define

$$\underline{\underline{G_B}}(X)$$

to consist of the quotients $q^*\tilde{M}/\mathcal{N}$ which are free with basis $\{x_b \,|\, b \in B\}$. Then we saw this was represented by an explicit affine scheme $G_B = \operatorname{Spec} H$. The idea is that we introduce variables a_{ib} such that

$$x_i = \sum_b a_{ib} x_b$$

and then

$$H = R\left[a_{ib} \,|\, i \in I, b \in B\right] / \left(a_{ib} = \delta_{ib}, i \in B; \sum_i r_{ij} a_{ib}, j \in J, b \in B\right)$$

is just the polynomial ring modulo these relations. The point is that this corresponds to the matrix

$$\begin{pmatrix} \mathrm{id}_{r\times r} & a_{ib} \end{pmatrix}$$

as in the classical case.

1.2. **Covering with open subfunctors cont'd.** Now we show that G_B is an open subfunctor. Recall this means the following. Given a functorial map $\underline{\underline{X}} \to \underline{\underline{G(\mathcal{M}, r)}}$ (or by Yoneda, given an element $q^*\mathcal{M}/\mathcal{N}$ on X) then there should correspond an open subset of X which is the preimage of $\underline{\underline{G_B}}$ under this functorial map. In other words, given $\varphi : T \to X$ (i.e. an element of $\underline{\underline{X}}(T)$) when is it in the preimage $\underline{\underline{G_B}}(T)$? Well this is in the preimage of $\underline{\underline{G_B}}(X)$ if it gives

$$\varphi^*\left(q^*\mathcal{M}/\mathcal{N}\right)$$

a free $\mathcal{O}_T$ module on basis B.

Write $\mathcal{V} = q^*\mathcal{M}/\mathcal{N}$. Locally we have opens $U \subseteq X$ on which $\mathcal{V}$ has some basis $v_1, \ldots, v_r \in \mathcal{V}(U)$. In terms of this basis, what is the condition for the x_bs to also be a matrix. These will each be a unique combination of these v_i with coefficients which are functions on U, i.e. we have an $r \times r$ matrix g over $\mathcal{O}(U)$ relating these. Then in this language, g is invertible iff the x_bs form a basis. Now we can consider its determinant $\det(g) \in \mathcal{O}(U)$ which has some vanishing locus, and its complement $U_{\det(g)}$ which is the largest open subset in which the x_b form a basis. This works for all choices of U. And now the claim is that for two different choices U and U', in the intersection $U_{\det(g)} \cap U'_{\det(g')}$ is still the largest open subset in the union where the x_b still form a basis and this is well defined.

So we get some $W \subseteq X$ (which depends on $\mathcal{V}$ of course) which is the largest open subset in V for which the x_B form a basis of $\mathcal{V}$. Then the claim is that $\underline{\underline{W}}$ is the preimage of $\underline{\underline{G_B}}$ in $\underline{\underline{X}}(T)$

Consider

$$\begin{array}{ccc} X & \xrightarrow{\quad r \quad} & G(\mathcal{M}, r) \\ & {\scriptstyle q}\searrow \quad \swarrow{\scriptstyle p} & \\ & S & \end{array} .$$

On $G(\mathcal{M}, r)$ we have tautological $\mathcal{V} = p^*\mathcal{M}/\mathcal{N}$ locally free of rank r. Now we can take the exterior power $\wedge^t \mathcal{V}$. This is locally free of rank 1, i.e. an invertible sheaf and we will write it as $\mathcal{L} = \wedge^t \mathcal{V}$. The idea is that this is ample as long as $\mathcal{M}$ is locally finitely generated. In fact it is even very ample. Now consider

$$p^* \wedge^r \mathcal{M} = \wedge^r p^* \mathcal{M} \twoheadrightarrow \mathcal{L}$$

and then we have the following global section in $p^* \wedge^r \mathcal{M}$ mapping to s:

$$x_b = x_{b_1} \wedge \ldots \wedge x_{b_r} \mapsto s \in \mathcal{L}(G(\mathcal{M}, r))$$

(or just in $\mathcal{L}(X)$) and then basically what the above calculation says is that

$$G(\mathcal{M}, r)_s = G_B \ .$$

But we don't have to assume $\mathcal{M}$ is locally finitely generated, we might as well do it more generally. From the above picture we get

$$\begin{array}{ccc} G(\mathcal{M}, r) & \xrightarrow{\quad j \quad} & \mathbb{P}(\wedge^r \mathcal{M}) = \operatorname{Proj} \mathcal{S}(\wedge^r \mathcal{M}) \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

which is somehow built into the story in the sense that it comes from $\mathcal{L}$. Note that j is such that $\mathcal{L} = j^*\mathcal{O}(1)$. Then the Proj is covered by the Y_{x_B}, and then

$$X_s = G_B = j^{-1}(Y_{x_B}) \ .$$

The point is we have an explicit description of this on affines, i.e.

$$\operatorname{Spec} H = X_f \to Y_{x_b}$$

which corresponds to a map $A \to H$ where generators of A look like x'_B/x_B and the variables of H look like these matrices

$$\begin{pmatrix} \mathrm{id}_{r \times r} & a_{ib} \end{pmatrix}$$

and these x'_B/x_B look exactly like the determinant of the minors. In other words this map $A \twoheadrightarrow H$ is surjective, which means

$$X_s \to Y_{x_B}$$

is a closed embedding, and the defining equations are given by the Plücker relations. And none of this depended on $\mathcal{M}$ being finitely generated.

So now we have that

$$G(\mathcal{M}, r) \hookrightarrow \mathbb{P}(\wedge^r \mathcal{M})$$

is a closed embedding, and it is defined by an ideal in $S(\wedge^r \mathcal{M})$ which is generated by the Plücker relations. What this says, is that $\mathcal{L} = \wedge^r \mathcal{V}$ is very ample. So if $\mathcal{M}$ is not finitely generated this (by definition) can't be ample, but it would still be very ample. But if $\mathcal{M}$ is locally finitely generated then this is of course ample, and in this case $p : G(\mathcal{M}, r) \to S$ is projective over S.

If $\mathcal{M} = \mathcal{O}_S^n$, is free of rank n, then it doesn't actually matter what S is in the sense that $G(\mathcal{M}, r)_S = G(M, r)_{\mathbb{Z}}$, and in this case we have the Plücker relations and no other relations, since $\mathcal{M}$ itself has no other relations, i.e. $S(\wedge^r \mathcal{M})$ is literally just a polynomial ring in these relations, and then

$$G(\mathcal{M}, r)_S = G(n, r)$$

is exactly the classical Grassmannian.

We will do some nice applications and examples next time.