LECTURE 31
MATH 256B

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1. Classical Grassmannians

Fix a ground field $k = \overline{k}$. Consider the classical Grassmannian $G(n,m) \xrightarrow{p} k$. Recall we want to think of this as parameterizing the quotients, i.e. the closed points are its k-valued points which parameterize $W \subseteq k^n$ such that

$$\dim\left(k^n/W\right) = m \ .$$

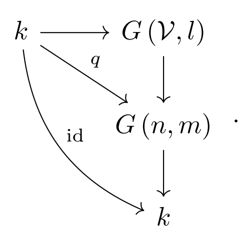
On $G(n,m)$ we have a tautological vector bundle

$$\mathcal{V} = \mathcal{O}^n/\mathcal{N}$$

locally free of rank m. $p^* k^n$ consists of sections of a rank m vector bundle (or dual of it). Then $V = \operatorname{Spec} S(\mathcal{V})$ is an algebraic vector bundle over $G(n,m)$. Now for $l \leq m$ we want to look at

$$\begin{array}{c} G(\mathcal{V}, l) \\ \downarrow \\ G(n,m) \end{array} \ .$$

This gadget will be like a flag variety. Let's see what it means to give a k-point of this:



Well we have $\dim(W) = n - m$, and for $W' \supseteq W$ with $\dim k^n/W' = l$, we want to think of this as $\left(k^n/W\right)/\left(W'/W\right)$. So the idea is that this parameterizes chains

$$k^n \supset W' \supset W \supset 0$$

where W' is of dimension $n - l$ and W is of dimension $n - m$. So this is a partial flag variety. If it had elements of every dimension it would be a complete flag.

One thing we can see from this is the following. We have a map $G(\mathcal{V}, l) \to G(n,l)$ which sends this sort of flag to the point W'. In terms of this construction this is the following. For any base S, and two sheaves $\mathcal{M}_1 \twoheadrightarrow \mathcal{M}_2$ and we can form $G(\mathcal{M}_1, r)$ and $G(\mathcal{M}_2, r)$ and then we have a natural map

$$G(\mathcal{M}_1, r) \leftarrow G(\mathcal{M}_2, r) \ .$$

In particular, on any scheme S we can wor with $\mathcal{O}^n$, and then

$$G\left(\mathcal{O}^n, l\right) = S \times_{\operatorname{Spec}\mathbb{Z}} G\left(\widetilde{\mathbb{Z}^n}, l\right)_{\mathbb{Z}} \ .$$

So in this case this is really a map

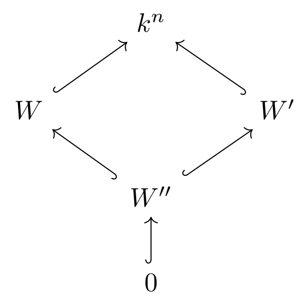
$$G\left(\mathcal{V}, l\right) \to G\left(n, l\right) \times G\left(n, m\right)$$

and this takes the chain and maps it to the pair (W', W). In fact this is a closed embedding.

Before we claimed the following. If we look at the locus in $G(n,l) \times G(n,m)$ where we have a pair

$$\left\{(W', W) \mid \dim\left(W' \cap W\right) \geq r\right\} \subseteq G\left(n, l\right) \times G\left(n, m\right)$$

then this looks like



where $\dim\left(W''\right) = r$, $\dim\left(W\right) = n - m$, and $\dim\left(W'\right) = n - l$. Then this sits as a triple inside

$$G\left(n, l\right) \times G\left(n, m\right) \times G\left(n, n-r\right) \ .$$

2. A question

How big is the homogeneous coordinate ring R such that $G(n,m) = \operatorname{Proj} R$? This isn't really well-defined. Somehow R is supposed to be natural. So it's

$$k\left[x_I\right] / \left(\text{Plücker relations}\right)$$

where the x_I are the Plücker coordinates (i.e. there are $\binom{n}{m}$ of them). Then the question is the dimension $\dim_k R_d$ for all d. Another way of thinking about this is as follows. This sits as a subvariety in

$$G\left(n, m\right) \subseteq \mathbb{P}^{\binom{n}{m}-1} \ .$$

So this has some

$$\mathcal{L} = \mathcal{O}\left(1\right) = \wedge^m \mathcal{V} \ .$$

We know immediately that

$$R_0 = k \qquad \qquad \dim_k R_1 = \binom{n}{m}$$

and in fact it turns out that

$$R_1 = \Gamma\left(\mathcal{L}\right) \qquad \qquad R_d = \Gamma\left(\mathcal{L}^d\right)$$

since we have

$$\left(\wedge^m \mathcal{V}\right)^{\otimes d} \to R_d \qquad \qquad S^d\left(\wedge^m \mathcal{V}\right) \to R_d \ .$$

Now $\mathrm{GL}_n \circlearrowright k^n$, so it acts on the Grassmannian, so it acts on the R_d. In fact we have an algebraic representation $\mathrm{GL}_n \circlearrowright R_d$. For GL_n there is a classification of irreducible representations given by highest weight-modules, and this allows us to determine the dimension of any weight space of any representation. Then $S^d(\wedge^m \mathcal{V}) \to R_d$ has a lot of irreducible components, but this map somehow kills all but one of them, and somehow it gives us the biggest one. In particular

$$\wedge^m k^n \xrightarrow{\simeq} R_1$$

is the highest weight module

$$\wedge^m k^n = V_{\underbrace{1,\ldots,1}_{m},0,\ldots,0} \ .$$

Then

$$V_{1,\ldots,1,0,\ldots,0}^{\otimes d} = V_{d,\ldots,d,0,\ldots,0} \oplus \ldots$$

is an irreducible representation of GL_n whose highest weight is just d times the first highest weight. These have characters given by the Schur functions

$$\chi_{R_d} = S_{d,\ldots,d,0,\ldots,0}(x_1,\ldots,x_n) \ .$$

Recall a semi-standard Young-tableaux is an $m \times d$ array of numbers between 1 and n which have to increase weakly along the rows, and strongly along the columns. Then the number of such objects is the dimension of R_d. But these are the same as three-dimensional young diagrams, so the takeaway is that $\dim R_d$ is the number of three-dimensional Young diagrams inside a $d \times m \times (n-m)$ rectangular prism. This is clearly symmetric in m and n, and it is also symmetric in d.

A Grassmannian is really a partial flag variety with only one step. If we put a complete flag in here, the stabilizer consists of the upper triangular matrices which is a Borel subgroup. Then for a partial flag we get a parabolic subgroup P. On G/P there are various line bundles, and they come from taking each of the quotient spaces corresponding to the elements of the flags, take the highest exterior power:

$$\wedge^{d_i} W_i \mathcal{L}_i \ ,$$

where $\dim W_i = n - d_i$. These aren't individually ample, since they're the pullback of the natural ample line bundle on the product of Grassmannians pulled back. Now we can take the tensor product

$$\mathcal{L} = L_1^{\otimes r_1} \otimes \ldots \otimes L_n^{\otimes r_n}$$

of all of these. These are at least generated by their global sections. Now we can take the global sections on G/B of one of these $\mathcal{L}$s:

$$\Gamma_{G/P}(\mathcal{L})$$

and this always an irreducible GL_n representation. As it turns out taking the Euler characteristic of the sheaf cohomology somehow yields the Weyl character formula. Then the story for Grassmannians is a special case of this.