LECTURE 32
MATH 256B

LECTURE: PROFESSOR MARK HAIMAN
NOTES: JACKSON VAN DYKE

1. Dimension of algebraic varieties

1.1. Classical definition. The intuition is what we would assume.

Example 1. Consider the affine plane. Inside these we have varieties that look like curves, $V(f(x,y))$ for f non-constant, and these things are of course 1-dimensional. If f is reducible then it might have different components, but they are each 1-dimensional. Then we have points, which of course have dimension 0. Dimension is really sort of something about irreducible components. Eventually we will say that the dimension of something with multiple irreducible components is the max of the dimensions of the components.

The classical definition is as follows. For X an algebraic variety over $k = \overline{k}$ then $\mathcal{O}(X)$ is a domain, and $K(X)$ is the fraction field. Here we are thinking of this as a scheme $X = \operatorname{Spec}(\mathcal{O}(X))$. Then we have that

$$K(X) = \mathcal{O}_{X,x}$$

where $x = (0)$. Then the dimension of X is the transcendence degree of $K(x)/k$.

Example 2. For $X = \mathbb{A}^n$, $\mathcal{O}(X) = k[x_1, \ldots, x_n]$ and $K(X) = k(x_1, \ldots, x_n)$, and then the transcendence degree is n.

This is the definition used by the Italian school of classical algebraic geometry, and they sort of just assumed this definition works as it should. But it turns out to be hard to show this. So we will define it in a different way, and this definition will be a theorem.

1.2. Combinatorial dimension. Let X be any topological space. Then define

$$\dim(X) := \sup\{n \mid \exists Z_0 \subsetneq Z_1 \subsetneq \ldots \subsetneq Z_n \subseteq X \text{ irreducible closed }\} .$$

We can refine this by defining

$$\operatorname{Codim}(Z,X) := \sup\{n \mid \exists Z = Z_0 \subsetneq Z_1 \subsetneq \ldots \subsetneq Z_n \subseteq X \text{ irreducible closed }\} .$$

for Z irreducible and closed. Now we have a notion of local dimension at x:

$$\dim_x(X) := \operatorname{Codim}\left(\overline{\{x\}}, X\right) .$$

Then we have that

$$\dim(X) = \sup_{Z \subseteq \operatorname{Irr}(X)} \operatorname{Codim}(Z,X)$$

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and for X_i the irreducible components of X we have

$$\dim(X) = \sup(\dim(X_i)) \ .$$

It is also the case that the dimension is the supremum of the dimensions of the elements of an open cover. This actually isn't so obvious. Let $U \subseteq X$ open. Then consider a chain of closed irreducibles

$$Z_0 \subsetneq Z_1 \subsetneq \ldots \subsetneq Z_n$$

with $Z_0 \cap U \neq \emptyset$. Then clearly for any Z_{i+1} we have $Z_{i+1} \cap U \neq \emptyset$. Proceed by contradiction and suppose $Z_{i+1} \cap U = Z_i \cap U$. Then

$$Z_{i+1} = Z_i \cup (Z_{i+1} \setminus U)$$

but this means Z_{i+1} is a union of two irreducibles, so it cannot be irreducible so we have the desired contradiction. Therefore we have that

$$\operatorname{Codim}(Z \cap U, U) \geq \operatorname{Codim}(Z, X) \ .$$

But it also works the other way around. For any closed subset $Z \subseteq U$, then we can consider the chains of closed irreducibles in U starting at Z. The closures in X are still irreducible closed so we have

$$\operatorname{Codim}(Z, U) = \operatorname{Codim}\left(\overline{Z}, X\right) \ .$$

Then this also implies that if we have a cover $X = \cup U_\alpha$ then

$$\dim X = \sup \dim U_\alpha \ .$$

This isn't always met, but if every closed $Z \subseteq X$ contains a closed point, then it will also be true that

$$\dim(X) = \sup_{x \in X_{\mathrm{cl}}} \dim_x(X) \ .$$

Any affine scheme has this property (since all ideals are contained in a maximal ideal). Jacobson schemes have this property almost trivially. But we want dimension theory to work on local rings which are definitely not Jacobson. If X is quasicompact it has this property because of the following. So it is covered by finitely many affines, and each of the affines have a closed point. But they might not be closed in X. I.e. we might have $p_2 \in \overline{\{p_1\}}$. But then this point has to be in another affine because it can't be in the one we started with. We can repeat this, and we eventually run out of affines. Every finite dimensional X has this property as well. We will see that any sort of classical variety is finite dimensional. If X is locally Noetherian[1] then this is true as well.

One thing that is true is that every nonempty scheme has a locally closed point. This is easy. Just take a closed point of any open affine. The fact that $x \in X$ is locally closed means it is open in its closure, i.e. $\{x\}$ is open in $\overline{\{x\}}$. Then let $Z = \overline{\{x\}}$. This is the space of a scheme which is locally Noetherian. Now we claim that $\dim Z \leq 1$. Take some $z \in Z$. Then we claim the codimension is at most 1, i.e. $\dim_z Z \leq 1$. We have that

$$\dim_z Z = \dim \mathcal{O}_{Z,z}$$

where this is the Krull dimension. Write $A = \mathcal{O}_{Z,z}$. In general, for the Krull dimension, we have that

$$\dim A = \dim \operatorname{Spec} A \ .$$

[1] I.e. we can cover it with affines which are Spec of a Noetherian ring.

Now there exists $f \in A$ such that $V(f) = \{x\}$. Now we can take Z to be reduced so A is a Noetherian local domain, i.e. there is $f \neq 0$ in A such that every prime ideal $\mathfrak{p} \neq 0$ contains f. Then we claim that under this hypothesis we claim:

Claim 1 (Artin-Tate)**.** $\dim A \leq 1$.

Take any $g \in \mathfrak{m}$ in the maximal ideal. Then let $\mathfrak{p}$ be a minimal prime containing g, $\mathfrak{p} \supseteq (g)$. Then for $f \in \mathfrak{p}$, Krull's theorem implies that $\operatorname{ht} \mathfrak{p} = \dim A_{\mathfrak{p}} = 1$. But $\mathfrak{p}$ is also minimal containing f, which means every element of the maximal ideal belongs to one of the minimal primes containing f. So let $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ be these primes.[2] So we have that

$$\mathfrak{m} \subseteq \bigcup_i \mathfrak{p}_i$$

and by prime avoidance $\mathfrak{m}$ must be inside one of these minimal primes, so we are done.

Every scheme has a locally closed point, and now we see that if it is locally Noetherian, its closure has finite dimension, so it has a closed point which is closed in the entire scheme.

1.3. **Dimension theory of Noetherian local rings.** To get favorable qualities we basically need it to be Noetherian, so we will basically assume this from now on.

For A Noetherian then $\operatorname{Spec} A$ is Noetherian. Of course we define it to be Noetherian if it has the ascending chain condition (ACC) i.e. every infinite ascending chain of ideals cannot be strictly increasing. Then a space is Noetherian if there are never infinitely many strictly decreasing chains of closed subsets. All together for $\operatorname{Spec} A$ we get

- DCC for closed subsets
- ACC for open subsets
- every open subset is quasi-compact.

For X Noetherian this implies X has finitely many irreducible components (which implies A has finitely many minimal primes). If we say X has infinitely many components then whenever $X = Z_1 \cup Z_2$ then at least one of the Z_i has infinitely many irreducible components as well.

[2] In a Noetherian ring there are finitely many of these.