LECTURE 33
MATH 256B

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1. Dimension theory of Noetherian local rings

Let $(A, \mathfrak{m})$ be a Noetherian local ring. We can't get very far without this assumption.

1.1. Dimension zero.

If A has dimension 0 then this means $\mathfrak{m}$ is the only prime, i.e. $\operatorname{Spec} A = \{\mathfrak{m}\}$. This is equivalent to $\mathfrak{m} = \sqrt{0}$ which is equivalent to $\mathfrak{m}^d = 0$ for some d. Now M has finite length n. If there exists

$$0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$$

we have that

$$M_{i+1}/M_i = A/\mathfrak{m} \ .$$

So the fact that $\mathfrak{m}^d = 0$ implies that A has finite length which implies $l\left(A/\mathfrak{m}^d\right)$ is bounded, which implies $\mathfrak{m}^d = \mathfrak{m}^{d+1}$ for some d, so $\mathfrak{m}^d = 0$ by Nakayama's lemma. So these are all equivalent.

So we understand dimension 0 quite well.

1.2. Prime avoidance.

Let R be any ring with prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ and I any ideal.

Lemma 1. *$I \subseteq \mathfrak{p}_1 \cup \ldots \cup \mathfrak{p}_n$ implies $I \subseteq P_i$ for some i.*

Proof. WLOG we can assume there is no

$$P_i \subseteq \bigcup_{j \neq i} P_j \ .$$

So there exists some $x_i \in P_i$ such that $x_i \notin P_j$ for $j \neq i$. Then define

$$a_k = \prod_{i \neq k} x_j \ .$$

This is in P_i for all $i \neq k$ and not in P_k.

Now assume $I \not\subseteq P_i$ for all i. Then we have some $b_k \in I \setminus P_k$ and we can form

$$a_k b_k$$

which is in I and P_i for $i \neq k$, but not in P_k. Then we can form

$$x = \sum a_k b_k \in I \setminus \bigcup_i P_i$$

as desired. □

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1.3. Artin-Rees lemma.

Lemma 2. *Let A be Noetherian with an ideal $I \subseteq A$. Let M be a f.g. module with submodule N. Then there exists some m such that*

$$N \cap I^d M = I^{d-m} \left(N \cap I^m M\right)$$

for all $d \geq m$.

Proof. Form the algebra

$$A\left[tI\right] = A \oplus I \oplus I^2 \oplus \cdots \ .$$

This is finitely generated over a Noetherian ring so it is Noetherian. Now make a similar definition for M:

$$M\left[tI\right] = M \oplus IM \oplus I^2 M \oplus \cdots \ .$$

So now this is a graded $A\left[tI\right]$ module which is (finitely) generated by the generators of M. Then we have

$$M\left[tI\right] = \bigoplus_d I^d M \supseteq \bigoplus_d N \cap I^d M$$

which is an $A\left[tI\right]$ submodule. So let's say this is generated in degrees $d \leq m$ for some m. □

1.4. Krull's principal ideal theorem.

Let $(A, \mathfrak{m})$ be a Noetherian local ring.

Theorem 1 (Krull)**.** *If a prime $\mathfrak{p}$ is minimal among primes containing (f), then* $\operatorname{ht}(\mathfrak{p}) \leq 1$.

Geometrically this is talking about irreducible components.

Proof. Localizing at $\mathfrak{p}$, WLOG we can take $\mathfrak{p} = \mathfrak{m}$. Then we want to show $\dim A \leq 1$. I.e. if $(A, \mathfrak{m})$ is a Noetherian local domain, not a field, and $\mathfrak{m}$ is minimal over (f) then 0 and $\mathfrak{m}$ are the only primes. In other words we want to show for $t \neq 0$ $\dim A/(t) = 0$.

We have the sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (f)/\left(f^d\right) & \longrightarrow & A/\left(f^d\right) & \longrightarrow & A/(f) & \longrightarrow & 0 \\ & & \big\downarrow \simeq & & & & & & \\ & & A/\left(f^{d-1}\right) & & & & & & \end{array}$$

So we get that the lengths satisfy:

$$\underbrace{\ell\left(A/\left(f^d\right)\right)}_{=d \cdot r} = \ell\left(A/\left(f^{d-1}\right)\right) + \underbrace{\ell\left(A/(f)\right)}_{=r}$$

and then we have

$$0 \to (t)/(t) \cap \left(f^d\right) \to A/\left(f^d\right) \to A/\left(t, f^d\right) \to 0 \ .$$

Then this implies that for $d \geq m$ we have

$$(t) \cap \left(f^d\right) = f^{d-m} \cdot \left((t) \cap (f^m)\right)$$

so we have

$$\ell\left((t)/f^{d-m}(t)\right) = (d-m) \cdot r$$

which means

$$\ell\left((t)\,/\,(t)\cap\left(f^d\right)\right)\geq(d-m)\cdot r$$

so

$$\ell\left(A/\left(t,f^d\right)\right)\leq dr-(d_0m)\cdot r=m\cdot r$$

which implies

$$\dim\left(A/\,(t)\right)=0\ .$$

as desired. $\square$

1.5. **Consequences of Krull's theorem.** Now for $\dim A > 0$, pick $x_1 \in \mathfrak{m}$ and $x_1 \notin$ the minimal prime. Then the number of minimum primes in a Noetherian ring is finite

$$\dim\left(A/x_i\right)<\dim A\ .$$

Now repeating this and applying the ACC to the $(x_1,\ldots,x_k)$ we stop at some n where we have

$$\dim A/\left(x_1,\ldots,x_n\right)=0\ .$$

So we can reduce the dimension by at most 1. In fact, by Krull's theorem we lowered it by exactly 1. For $\mathfrak{m}$ minimal over $(x_1,\ldots,x_n)$ we have $\dim A/\left(\underline{x}\right)=0$ so $n\leq\dim A$ so in fact

$$\dim A=n\ .$$

Such a thing is called a *system of parameters*.