LECTURE 34
MATH 256B

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We will wrap up dimension theory so we can do some sheaf cohomology and smoothness.

1. Dimension theory

Let $f : X \to Y$ be finite. Then we know

(i) f having discrete fibers implies $\dim_x X \leq \dim_y Y$ where $y = f(x)$.

(ii) f satisfy the "going up theorem".[1] I.e. if we have $W \subseteq X$ and Z irreducible such that

$$\begin{array}{ccc} W & \hookrightarrow & X \\ \downarrow & & \downarrow f \\ Z & \hookrightarrow & Y \end{array}$$

then for any $Z' \subseteq Z$ irreducible, we can find $W' \subseteq W$ such that

$$\begin{array}{ccccc} W' & \dashrightarrow & W & \hookrightarrow & X \\ \downarrow & & \downarrow & & \downarrow f \\ Z' & \hookrightarrow & Z & \hookrightarrow & Y \end{array} .$$

(iii) We also have the going down theorem which says the following.

Theorem 1 (Going down)**.** *Let $f : X \to Y$ be finite and surjective. Let $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$ be affine irreducible. If A is integrally closed, i.e. it is integrally closed in its field of fractions $K(A)$, then*

$$\begin{array}{ccccc} W' & \dashrightarrow & W & \dashrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ Z' & \hookrightarrow & Z & \hookrightarrow & Y \end{array} ,$$

i.e. for any $Z' \subseteq Z \subseteq Y$ irreducible and W' as before, then there is some W as in the above diagram.

Counterexample 1 (In the case A is not integrally closed)**.** Consider a nodal curve such as $C = V\left(y^2 - x^2(x+1)\right)$. This curve crosses itself at the origin and is irreducible. We can parameterize this curve with an affine line $\mathbb{A} = \operatorname{Spec} k[t]$. Explicitly this is

$$t \mapsto (x, y) = \left(t^2 - 1, t\left(t^2 - 1\right)\right) .$$

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[1] Geometrically this is the going down theorem.

Write the map $g : \mathbb{A}^1 \to X$. In this case $t = \pm 1$ both map to the origin. This corresponds to a map

$$k[x,y] / \left(y^2 - x^2(x+1)\right) \hookrightarrow k[t] \ .$$

This is a finite morphism, and $k[t]$ is integrally closed, and these rings have the same field of fractions which is $k(t)$. So this is a situation where these conditions are not met.

Now take $Y = \mathbb{A}^2 \times C$ and then $X = \mathbb{A}^1 \times \mathbb{A}^2$ the parameterization of Let z be the graph of g. Now take $z' \in Y$ a point on the nodal curve on z. Then there is a unique life of z to $\mathbb{A}^1 \times \mathbb{A}^2$.

When going down holds, then it will be true that X and Y are the same dimension (which we new already from going up) but in fact it will hold point-wise:

$$\dim_x X = \dim_y Y \ .$$

This is good for the following reason. Let X be an irreducible algebraic scheme over k. Then we have the following theorem.

Theorem 2. $\dim_x X = \operatorname{tr}\deg_k K(X)$ *for all* $x \in X_\ell$.

Noether's normalization lemma will give us that there exists a finite surjective morphism $X \to \mathbb{A}^d_k = \operatorname{Spec} k[t_1, \dots, t_d]$ such that $x \mapsto m = (\underline{t})$ is the origin.

The point is that going down down applies to give us

$$\dim_x X = \dim_m \mathbb{A}^d = \dim [\underline{t}]_m = d = \operatorname{tr}\deg_k K(X) \ .$$

1.1. **Noether normalization.** Let $R = k[\underline{x}]/I$ and $n \subseteq R$. Then we want $t_1, \dots, t_d \in n$ algebraically independent, $k[t_1, \dots, t_d] \hookrightarrow R$, and R is a finitely generated $k[\underline{t}]$ module.

Step 1. First we do it without the algebraically independent condition. So we have R/n finite over k. Let $a_1, \dots, a_r$ span R/n. Now write

$$x_i = w_i + \sum d_i a_i$$

where $w_i \in n$ and

$$a_i a_j = z_{ij} + \sum c_k a_j$$

for $z_{ij} \in n$. Then $\underline{a}$, $\underline{w}$, $\underline{z}$ generate R/n. Then we have that R is finitely generated over $k[\underline{w}, \underline{z}]$. I.e. if the ws and zs were algebraically independent then we would be done.

Step 2. So we have R a finitely generated module over $k[t_1, \dots, t_m]$ not algebraically independent. Then suppose $k[t_1, \dots, t_n] \ni f(t_1, \dots, t_n) = 0$ in R. Now we do some sort of change of variables. Let $u_n = t_n$ and

$$u_n = t_i - t_n^{N_i}$$

for $0 \ll N_1 \ll \dots$. Then we have that

$$t_i \mapsto u_i + u_n^{N_i}$$

is monic in the u_n. Now we can just remove one of the variables. Now we can repeat this process until they are algebraically independent.