# LECTURE 35
# MATH 256B

LECTURE: PROFESSOR MARK HAIMAN
NOTES: JACKSON VAN DYKE

## 1. Finishing up dimension theory

We saw the following last time. For $X$ an irreducible algebraic scheme over $k$, then we get that the local dimension

$$\dim_x X = \operatorname{tr}\deg_k K(x)$$

for all $x \in X_\ell$. It has the following consequences.

1.1. **Catenariness.** Consider a Noetherian local ring $(A, \mathfrak{m})$. Consider some saturated chain (which is finite since $A$ is Noetherian) $Q_0 = \mathfrak{m} \supsetneq Q_1 \supsetneq \ldots \supsetneq Q_d = \mathfrak{p}$ where $d$ is the dimension, i.e. $\mathfrak{p}$ is minimal. It is possible that we can have another saturated chain of a different length. It is certainly bounded by $d$, but in fact it can be strictly less, i.e. for $\mathfrak{p}'$ in the shorter chain we have

$$\operatorname{ht}(\mathfrak{p}') + \dim A/\mathfrak{p}' \lneq \dim A \ .$$

If this doesn't happen $A$ is called *catenary*.

Let $z \in Z_{\mathrm{cl}} \subseteq \mathbb{A}_k^n$ be irreducible and closed. Then

$$\underbrace{\dim_z Z}_{=\dim Z} + \underbrace{\operatorname{Codim}(z, \mathbb{A}^n)}_{\dim_p \mathbb{A}^n} = n$$

where $Z = \overline{\{p\}}$. It is automatic that

$$\dim_z Z + \operatorname{Codim}(z, \mathbb{A}^n) \leq n \ .$$

Let $f_1, \ldots, f_k \in \mathfrak{p}$ be a s.o.p. in $k[t]_{\mathfrak{p}}$, i.e. $Z$ is an irreducible component of $V(f)$ and $k = \operatorname{Codim}(Z, \mathbb{A}^n)$.

Let $x \in Z_{\mathrm{cl}}$ not in any other component of $V(\underline{f})$. Let $g_1, \ldots, f_l$ be an s.e.p. in $\mathcal{O}(Z)_x = (k[\underline{t}]/\mathfrak{p})_x$. Then $l = \dim_x Z = \dim_z Z = \dim Z$ and $V(\underline{f}, \underline{g}) = \{x\}$ in $k[\underline{t}]_x$ which implies $b + l \geq n$.

1.2. **Two example.** One might want to know when something is irreducible and the dimension of its components etc. One can often find this out by playing with number of equations and open sets where it is obvious. Consider

$$V(x_1 y_1 + \cdots + x_n y_n) \subseteq \mathbb{A}^n \times \mathbb{A}^n \ .$$

There is no mystery about its dimension. By Krull's theorem this is dimension at least $n-1$, but $\mathbb{A}^n$ is irreducible, so this is certainly of dimension $n-1$. We want to see that it is irreducible. The map

$$\pi : X \to \mathbb{A}^n$$

*Date*: April 24, 2019.

has fibers $\mathbb{A}^{n-1}$ except

$$U = \pi^{-1}\left(\mathbb{A}^n \setminus 0\right)$$

which is obviously irreducible of dimension $2n-1$. Then

$$X = \pi^{-1}(0) \cup \overline{U}$$

so $X$ must be irreducible.

This same game of counting equations and looking at fibers is useful in general.

## 2. Flatness and fiber dimensions

Consider $f : X \to Y$, local Noetherian, and $x \in Z \subseteq X$. Then

$$\dim_x Z + \dim_y Y \geq \dim_x X \ .$$

**Claim 1.** This is an equality if $f$ is a flat morphism.

Flatness somehow says that fibers vary in a continuous manner.

**Lemma 1.** *Consider a local homomorphism* $(A, \mathfrak{m}) \to (B, \mathfrak{n})$. *Let* $B$ *be a flat* $A$ *module. Then* $\operatorname{Spec} B \to \operatorname{Spec} A$ *is surjective.*

The thing to think about here is $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$. Geometrically the thing going on here is that flatness implies going down. Then going down implies that

$$\dim_x f^{-1}(y) + \dim_y Y \leq \dim_x X \ .$$

Therefore, the general fiber dimension inequality from before implies that in the local Noetherian case we get equality.

Consider the product of two schemes $X$ and $Y$ over $k$. So this fits in the diagram

$$\begin{array}{ccc} (X \times_k Y) & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & \operatorname{Spec} k \end{array} \ .$$

Everything is flat in this picture. Take some $(x, y) \in (X \times_k Y)_{\mathrm{cl}}$. Then let $Z = p_X^{-1}(x)$ and we have

$$\begin{array}{ccc} X & \longrightarrow & \operatorname{Spec} k(x) \\ \big\downarrow{\scriptstyle \text{finite}} & & \big\downarrow{\scriptstyle \text{finite}} \\ Y & \longrightarrow & \operatorname{Spec} k \end{array}$$

and we get

$$\dim_{(x,y)} X \times_k Y = \underbrace{\dim_{(x,y)} Z}_{\dim_y Y} + \dim_x X$$

as we might suspect.

For any field $k$ consider

$$\begin{array}{ccc} \operatorname{Spec} k(x) \otimes_k k(y) & \longrightarrow & \operatorname{Spec} k(x) \\ \downarrow & & \downarrow \\ \operatorname{Spec} k(y) & \longrightarrow & \operatorname{Spec} k \end{array} \ .$$

Then for the map

$$k(x) \otimes_k k(y) \to k(x, y)$$

the image is not a field and the kernel is prime, so the kernel is not maximal which means the dimension of $\operatorname{Spec} k(x) \otimes_k k(y)$ is actually equal to 1.