LECTURE 36
MATH 256B

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1. Cohomology

1.1. **Usual cohomology.** Let A be an abelian group. What do we mean by the cohomology groups $H^i(X, A)$ of a space X. If X is a smooth real manifold and we want $A = \mathbb{R}$ or $\mathbb{C}$, then we can get this with de Rham cohomology. I.e. we can look at the k-forms, or k-cochains, then then $\partial = d$ maps these to the $k+1$ cochains. Of course $d^2 = 0$, a form is called closed if the differential of it is 0, and if it is in the image of d it is called exact. Then the closed forms modulo the exact forms form the de Rham cohomology.

For much more general spaces and coefficients, we can construct simplicial cohomology. In this case a k-cochain is a function of k-dimensional simplices $\sigma : \Delta_k \to X$ to A:

$$f : \{\sigma : \Delta_k \to X\} \to A \ .$$

Then the boundary map ∂ takes k-simplices to $k+1$ simplices and is defined by:

$$(\partial f)(\sigma) = \sum \pm f(\sigma_i) \ .$$

where $\sigma : \Delta_{k+1} \to X$. I.e. it takes the sum of the values of f on the k-dimensional faces (simplices) of the $k+1$-dimensional simplex. $\partial^2 = 0$ as before, and then at each stage we can look at the cycles and mod out by the boundaries to get the ith cohomology. This coincides with the de Rham cohomology when X is a manifold, but even this doesn't work for arbitrary spaces.

Since we want to work in the Zariski topology, we need something even more general than this. In our situation we want to allow for coefficients in some quasi-coherent sheaf.

1.2. **Sheaf cohomology.** So we let X be any space whatsoever, and $\mathcal{A}$ a sheaf of abelian groups and we want to define sheaf cohomology $H^i(X, \mathcal{A})$. In particular we define it to be the ith derived functor

$$R^i\Gamma(X, \mathcal{A}) \ .$$

This reduces to the usual cohomology when $\mathcal{A} = \underline{A}$ is the constant sheaf.

$R^0\Gamma(X, \underline{A})$ consists of functions from A to the connected components of X. This is what we get in the usual cohomology theories as well.

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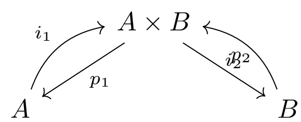
2. Homological algebra

2.1. Abelian categories.

Example 1. Modules over any ring (even noncommutative), sheaves of $\mathcal{O}_X$-modules, and the category of complexes are all examples. We will primarily be interested in the second example.

One common feature between all of these examples is that $\operatorname{Hom}(A, B)$ forms an abelian group, and composition is bilinear. If we just have this, the category is sometimes called *pre-additive*.

But it turns out something nice happens in this situation. Assume our category has a product. Then if it's an abelian category we can construct the maps $i_1 = (1_A, 0)$ and $i_2 = (0, 1_B)$:



In any category when we have such four morphisms we have a product, and in fact these are automatically coproducts. We should think of this as a direct sum.

If we assume that these all exist then it is called an *additive* category, i.e. if it has finite direct sums.

Now consider a map $f : A \to B$. This might have a kernel, which is an object K and a morphism $i : K \to A$ which is terminal among maps $K \to A$ which compose with f to be trivial. Similarly it might have a cokernel, which is an object Q and a morphism $q : B \to Q$ which is universal with respect to this property.

The cokernel of i is called the coimage of f, and the kernel of the cokernel is called the image of f. We always have a map $\operatorname{coim}(f) \to \operatorname{im}(f)$, and for modules this is an isomorphism. We will call the category *abelian* if it has kernels and cokernels, and that coimages are always the same as images.

There is a theorem which says that any abelian category is the same as a full subcategory of the category of modules over some ring. So we know that we can stop specifying properties of this category. The moral of this is that it's okay to pretend we're working with modules when we're really working an abelian category. One reason is this general representability theorem, the other is that we can use a more functorial approach (looking at functors representing objects instead) and the third is to do things sort of stalk by stalk.

2.2. Complexes.

So let $\mathcal{A}$ be any abelian category. Then $\mathcal{C}(\mathcal{A})$ consists of *complexes* of $\mathcal{A}$. These are collections of objects A^i for $i \in \mathbb{Z}$ and maps ∂_i:

$$\cdots \xrightarrow{\partial_{i-2}} A^{i-1} \xrightarrow{\partial_{i-1}} A^i \xrightarrow{\partial_i} A^{i+1} \xrightarrow{\partial_{i+1}} \cdots$$

Then a morphism is a collection of maps $f_i : A^i \to B^i$ such that all of the squares commute:

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{\partial_{i-2}} & A^{i-1} & \xrightarrow{\partial_{i-1}} & A^i & \xrightarrow{\partial_i} & A^{i+1} & \xrightarrow{\partial_{i+1}} & \cdots \\
\downarrow{\scriptstyle f_{i-2}} & & \downarrow{\scriptstyle f_{i-1}} & & \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle f_{i+1}} & & \\
\cdots & \xrightarrow{\partial_{i-2}} & B^{i-1} & \xrightarrow{\partial_{i-1}} & B^i & \xrightarrow{\partial_i} & B^{i+1} & \xrightarrow{\partial_{i+1}} & \cdots
\end{array}$$

Any such complex has cohomology:

$$H^i(A^\bullet) = \ker \partial_i / \operatorname{im} \partial_{i-1} \ .$$

Definition 1. $f : A^\bullet \to B^\bullet$ is a *quasi-isomorphism* if it induces an isomorphism $H^i(A^\bullet) \xrightarrow{\cong} H^i(B^\bullet)$ for all i.

Of course every quasi-isomorphism is not an isomorphism.

Counterexample 1. If we have the complex

$$\ldots \to 0 \to A \xrightarrow{1_A} A \to 0 \to \ldots$$

then this has homology 0, so it is quasi-isomorphic to the zero complex, but of course it isn't isomorphic to this complex.

The derived category is

$$\mathcal{D}(\mathcal{A}) = Q^{-1}\mathcal{C}(\mathcal{A})$$

where we have inverted all of the quasi-isomorphisms.

Remark 1. There are some set-theoretic issues when forming this, but as it turns out the right strategy for dealing with them is ignoring them.

Consider an exact sequence of complexes:

$$0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0 \ .$$

This induces a long-exact sequence

$$\ldots \to H^{i-1}(C) \to H^i(A) \to H^i(B) \to H^i(C) \to H^{i+1} \to H^{i+1}(B) \to \ldots$$

which we will see next time.