LECTURE 37
MATH 256B

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1. Long-exact sequences

Fix an abelian category $\mathcal{A}$ and consider the category of complexes $\mathbf{C}(\mathcal{A})$. Let $f : A \to B$ be a morphism in this category. Then we can define the *mapping cone of* f, $\operatorname{cone}(f)$. In the ith degree we put A^{i+1} and B^i. Then the maps in this complex are defined as follows:

$$\begin{array}{ccc} A^i & \xrightarrow{-\partial} & A^{i+1} \\ \oplus & \searrow^{f} & \oplus \\ B^{i-1} & \xrightarrow{\partial} & B^i \\ & & \\ (i-1) & & (i) \end{array} \quad .$$

Remark 1. The situation with signs here is that the shift operator and the boundary sort of anti-commute. If you follow this generic rule signs should always work out.

We have a canonical injective map $B \to \operatorname{cone}(f)$ and the additional maps:

$$A \xrightarrow{f} B \longrightarrow \operatorname{cone}(f) \longrightarrow A[1] \xrightarrow{f} B[1] \quad .$$

This goes on forever, and is called an *exact triangle* and is written either as:

$$\begin{array}{ccc} A & \longrightarrow & B \\ {}_{+1}\nwarrow & & \swarrow \\ & \operatorname{cone}(f) & \end{array}$$

or as

$$A \to B \to \operatorname{cone}(f) \xrightarrow{+1} \quad .$$

The point here is that such things induce an exact sequence on H^i.

Example 1. We show exactness at the step $B \to \operatorname{cone}(f) \to A$. Take a cycle $(a, b) \in Z^i(\operatorname{cone}(f))$. Recall this means $\partial a = 0$ and $f(a) + \partial b = 0$. Saying this maps to 0 in A means $a = \partial a'$ for some a'. So we want to show (a, b) is in the image of the cohomology, i.e. we want to find a cycle in B and boundary in $\operatorname{cone}(f)$ which is equal to (a, b) modulo boundaries.

$f(a) + \partial b = 0$ means $\partial(f(a') + b) = 0$, so $f(a') + b \in B^i$ is a cycle so it maps to $(0, f(a') + b)$. Now we have

$$\partial(a', 0) = (-a, f(a'))$$

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so we have that

$$(a,b) = (0, f(a') + b) - \partial(a', 0)$$

and we are done.

Lemma 1. *Consider some subcomplex $N \subseteq A$.*

(i) If N is acyclic[1] then $A \xrightarrow{\simeq} A/N$ is a quasi-isomorphism.
(ii) Dually, if the quotient is acyclic, then $N \overset{\simeq}{\hookrightarrow} A$ is a quasi-isomorphism.

Proof. We only prove the first statement. Let N be acyclic. Let $x \in A^i$ with $\partial x \in N^{i+1}$. Then there exists some $y \in N^i$ such that $\partial x = \partial y$. Then $\partial(x-y) = 0$, which means $x - y \in Z^i(A)$. ☐

Let

$$A \xrightarrow{f} B \to \operatorname{cone}(f) \xrightarrow{+1}$$

be an exact triangle. Then this induces a long exact sequence of objects of $\mathcal{A}$:

$$\cdots \to H^i(A) \to H^i(B) \to H^i \operatorname{cone}(f) \to H^{i+1}(A) \to \cdots .$$

Then we have:

Corollary 1. *f is a quasi-isomorphism iff $\operatorname{cone}(f)$ is acyclic.*

Let $f : A \to B$. Then we have an exact sequence:

$$0 \longrightarrow \ker(f)[1] \longrightarrow \operatorname{cone}(f) \longrightarrow \operatorname{cone}(\in f \to B) \longrightarrow 0 \quad .$$

Now suppose f is surjective. Then the cone of the isomorphism $\operatorname{cone}(\operatorname{im} f \to B)$ is acyclic, which means $\ker(f) \xrightarrow{\simeq} \operatorname{cone}(f)[-1]$ is a quasi-isomorphism.

So if f is surjective the picture is

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker f & \longrightarrow & A & \xrightarrow{f} & B & \longrightarrow & 0 \\ & & \downarrow{\simeq} & & & & & & \\ & & \operatorname{cone}(f)[-1] & & & & & & \end{array} .$$

Now consider a SES:

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 .$$

Then $A \simeq \operatorname{cone}(f)[-1]$. There is also

$$0 \longrightarrow A \oplus \operatorname{im} f \longrightarrow \operatorname{cone}(f) \longrightarrow \operatorname{coker} f \longrightarrow 0 \quad .$$

which is somehow dual to the other picture. This sequence is the same as:

$$0 \longrightarrow \operatorname{cone}(A \to \operatorname{im} f) \longrightarrow \operatorname{cone}(f) \longrightarrow \operatorname{coker} f \longrightarrow 0 \quad .$$

Now if f is injective then this means

$$A \simeq \operatorname{im} f$$

is a quasi-isomorphism, and

$$\operatorname{cone}(f) \simeq \operatorname{coker} f$$

is a quasi-isomorphism.

[1]This means all cohomology is trivial.

So we have two ways of somehow turning a SES into a triangle:

$$A \longrightarrow B \longrightarrow \operatorname{cone}(f) \xrightarrow{+1} \dots$$

$$\operatorname{cone}(g)[-1] \longrightarrow A \longrightarrow B \xrightarrow{+1} \dots$$

but we can rewrite the second as:

$$B \longrightarrow C \longrightarrow \operatorname{cone}(g) = A[1] \xrightarrow{+1} \dots \quad .$$

Now we can compare these as follows. We have the following maps:

$$\begin{array}{ccc} C & \underset{\simeq}{\longleftarrow} & \operatorname{cone}(f) \\ \downarrow & & \downarrow \\ \operatorname{cone}(g) & \underset{\simeq}{\longleftarrow} & A[1] \end{array}$$

where $\simeq$ denotes quasi-isomorphism. This diagram homotopy anti-commutes. The point is that we get two maps $C \to A[1]$ in the derived category by following the two sides of this diagram.

So for an exact sequence of complexes

$$0 \to A \to B \to C \to 0$$

we get the long exact sequence

$$\cdots \to H^i(A) \to H^i(B) \to H^i(C) \to H^{i-1}(A) \to \cdots \quad .$$

2. Homotopic maps

Consider two complexes $A^\bullet$ and $B^\bullet$. Then we can form another complex $\operatorname{Hom}^\bullet(A,B)$ which has

$$\operatorname{Hom}^i(A,B) = \prod_j \operatorname{Hom}\left(A^j, B^{j+i}\right) \quad .$$

For some j and some $\varphi : A^j \to B^{j+i}$ we have the diagram:

$$\begin{array}{ccc} A^j & \xrightarrow{\varphi} & B^{j+i} \\ \partial\uparrow & \overset{\varphi\partial}{\nearrow} & \\ A^{j-1} & & \end{array}$$

but we can also consider

$$\begin{array}{ccc} & & B^{j+i+1} \\ & \overset{\partial\varphi}{\nearrow} & \partial\uparrow \\ A^j & \xrightarrow{\varphi} & B^{j+i} \end{array} \quad .$$

Then the boundary map in the complex $\operatorname{Hom}^\bullet(A,B)$ is given by:

$$\partial_{\operatorname{Hom}} = \varphi\partial - (-1)^i \partial\varphi \ .$$

We can check this squares to 0.

Consider $Z^0 (\mathrm{Hom}^\bullet)(A, B)$. This consists of maps $A^i \to B^i$ for every i. Then the fact that these are cycles means exactly that the following diagram commutes:

$$\begin{array}{ccc} A^i & \xrightarrow{\partial} & A^{i+1} \\ \downarrow \varphi & & \downarrow \varphi \\ B^i & \xrightarrow{\partial} & B^{i+1} \end{array} .$$

But this is exactly what it means to be a morphism of complexes, so

$$Z^0 \,\mathrm{Hom}^\bullet (A, B) = \mathrm{Hom}(A, B) \ .$$

Then the 0-boundaries $B^0 \,\mathrm{Hom}^\bullet (A, B)$ are maps $A^i \to B^i$ of the form $f = \partial s + s\partial$ for $s : A^i \to B^{i-1}$ such that the following diagram commutes:

$$\begin{array}{ccccc} & & A^i & \xrightarrow{\partial} & A^{i+1} \\ & \swarrow s & & \swarrow s & \\ B^{i-1} & \xrightarrow{\partial} & B^i & & \end{array} .$$

We say such maps are *homotopic to the identity* and write $f \sim 0$.

Fact 1. *Let $f : A \to B$. Then TFAE:*

(i) $f \sim 0$
(ii) f *factors as* $A \to \mathrm{cone}(1_A) \to B$
(iii) f *factors as* $A \to \mathrm{cone}(1_B)[-1] \to B$

This means if it is homotopic to 0 it induces the zero arrow in the derived category so the functor $\mathbf{C}(\mathcal{A}) \to \mathbf{D}(\mathcal{A})$ factors as

$$\mathbf{C}(\mathcal{A}) \to \mathbf{K}(\mathcal{A}) \to \mathbf{D}(\mathcal{A}) \ .$$