LECTURE 38
MATH 256B

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1. Setup

Let $\mathbf{A}$ and $\mathbf{B}$ be abelian categories and $F : \mathbf{A} \to \mathbf{B}$ an additive functor. We should think of this as sheaves on a space and the global sections functor. This gives us a functor

$$F : \mathbf{C}(\mathbf{A}) \to \mathbf{C}(\mathbf{B})$$

between complexes in these categories. We also get a map on the categories of complexes up to homotopy:

$$F : \mathbf{K}(\mathbf{A}) \to \mathbf{K}(\mathbf{B}) \ .$$

It would be nice if we got a functor on the derived categories

$$\mathbf{D}(\mathbf{A}) \to \mathbf{D}(\mathbf{B})$$

but this only happens when F is exact. In fact we have the following diagram:

$$\begin{array}{ccc} \mathbf{C}(\mathbf{A}) & \xrightarrow{F} & \mathbf{C}(\mathbf{B}) \\ \downarrow & & \downarrow \\ \mathbf{K}(\mathbf{A}) & \xrightarrow{F} & \mathbf{K}(\mathbf{B}) \\ \downarrow & & \downarrow \\ \mathbf{D}(\mathbf{A}) & \xrightarrow{RF} & \mathbf{D}(\mathbf{B}) \end{array}$$

which won't exactly commute, but rather we want a map RF which will satisfy some universal property. Following Deligne we will construct this map.

2. Construction of RF

Consider some object $A \in \mathbf{C}(\mathbf{A})$ which is quasi-isomorphic to I. Then for $f : A \to B$ we can complete the diagram to:

$$\begin{array}{ccc} A & \xrightarrow{\simeq} & I \\ \downarrow f & & \downarrow g \\ B & \xrightarrow{\simeq} & J \end{array}$$

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and if f is a quasi-isomorphism so is g. This commutes up to homotopy. These form a filtered inductive system, i.e. we have

$$\begin{array}{ccc} A & \xrightarrow{\simeq} & I \\ \downarrow{\simeq} & & \downarrow{\simeq} \\ I' & \xrightarrow{\simeq} & I'' \end{array} .$$

In general the idea is the following. Let

$$F : \mathbf{K}(\mathbf{A}) \to \mathbf{D}$$

be any functor. Then we define

$$r_F(A)(-) : \mathbf{D}^{\mathrm{op}} \to \mathbf{Set}$$

to be

$$\varinjlim_{A \xrightarrow{\simeq} I} \operatorname{Hom}(-, F(I)) .$$

This is also functorial in A because we can complete diagrams.

Now we return to the case that $\mathbf{D} = \mathbf{D}(\mathbf{B})$. Now Deligne's definition is the following:

Definition 1. $RF(A) = X$ means

$$r_F(A)(-) = \operatorname{Hom}_{\mathbf{D}(\mathbf{B})}(-, X) .$$

This factors as:

$$\begin{array}{ccccc} & & \mathbf{K}(\mathbf{A}) & \xrightarrow{RF} & \mathbf{D}(\mathbf{B}) \\ & & \downarrow & \nearrow & \\ \mathbf{D}' & \subseteq & \mathbf{D}(\mathbf{A}) & & \end{array}$$

where in general this is really only defined on some $\mathbf{D}'$.

In fact, triangles in $\mathbf{D}(\mathbf{A})$ which come from mapping cones get send to triangles in $\mathbf{D}(\mathbf{B})$ which come from mapping cones, i.e. this is a triangle preserving functor.

3. Non-exact case

When F is not exact RF disagrees with F, so we can ask exactly what happens. We can do this for arbitrary F, but it is more interesting for left-exact F. Recall that for

$$0 \to A \xrightarrow{f} B \to C \to 0$$

we get

$$\cdots \to A \xrightarrow{f} B \to \operatorname{cone}(f) \xrightarrow{+1} \cdots .$$

Then we also get

$$\cdots \to RF(A) \to RF(B) \to RF(C) \xrightarrow{+1} \cdots .$$

We can also take homology, which is abbreviated

$$H^i RF(A) = R^i F(A)$$

to get a long-exact sequence:

$$\cdots \to R^i F(A) \to R^i F(B) \to R^i F(C) \to R^{i+1} F(A) \to \cdots .$$

If we have a complex which cohomology 0 below some point, then it is quasi-isomorphic to some complex which is actually 0 below some point. I.e. for $A \in \mathcal{A}$ we have

$$R^i F(A) = 0$$

for $i < 0$, where A is really the complex

$$\cdots \to 0 \to A \to 0 \to \cdots \ .$$

But in fact, if F is left exact we have that

$$R^0 F(A) = F(A) \ .$$

This means in the long-exact sequence we get

$$\cdots 0 \to F(A) \to F(B) \to F(C) \to R^1 F(A) \to \cdots \ .$$

4.

Suppose we have a class of objects $\mathcal{I} \subseteq \mathbf{C}(\mathbf{A})$. For $I, I' \in \mathcal{I}$ which are quasi-isomorphic $q : I \xrightarrow{\simeq} I'$ then we want to insist that F preserves this quasi-isomorphism

$$F(q) : F(A) \xrightarrow{\simeq} F(I') \ .$$

Assume $\mathcal{I}$ is cofinal with respect to the system $\left(A \xrightarrow{\simeq} I\right)$. Then

$$RF(A) = F(I)$$

for any $q : A \xrightarrow{q} I \in \mathcal{I}$.

Now suppose $\mathcal{J} \subseteq \mathbf{A}$ is

(i) closed under $\oplus$,
(ii) every $A \in \mathbf{A}$ has $0 \to A \to J \in \mathcal{J}$,
(iii) if $0 \to I \to J \to C \to 0$ and $I, J \in \mathcal{J}$ then $C \in \mathcal{J}$, and
(iv) if $0 \to I \to A \to B \to 0$ and $I \in \mathcal{J}$, then $0 \to F(I) \to F(A) \to F(B) \to 0$.

If this is the case, then $RF(A)$ is defined on $\mathbf{D}^+(\mathcal{A})$.

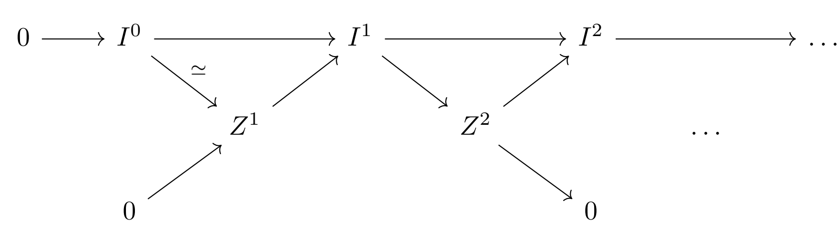
This means

$$A \xrightarrow{\simeq} I \in \mathbf{C}^+(\mathcal{J})$$

implies

$$RF(A) = F(I) \ .$$

Let $I, I \in \mathbf{C}^+(\mathcal{J})$ be quasi-isomorphic under q. Then we can take the mapping cone $\operatorname{cone}(q) \in \mathbf{C}^+(\mathcal{J})$. Then F preserves acyclic complexes. For an acylic complex $I^\bullet$ we have a bunch of short exact sequences:



then all of these little exact sequences remain exact under F. Then it turns out every complex in $\mathbf{D}^+(\mathbf{A})$ has a resolution of this form so it is indeed defined on $\mathbf{D}^+(\mathbf{A})$.

Example 1. If $\mathbf{A}$ has enough injectives then $\mathcal{J}$ consisting of the injectives satisfies the above properties. Modules and sheaves have this property.

Example 2. Let $F = \Gamma : \mathcal{O}_X\text{-}\mathbf{Mod} \to \mathcal{O}_X(X)\text{-}\mathbf{Mod}$. Then

$$H^i(X, \mathcal{L}) = R^i \Gamma_X(\mathcal{L}) \ .$$

We can instead look at Flasque sheaves. This means every section is the restriction of some global section, i.e. $\mathcal{F}(X) \twoheadrightarrow \mathcal{F}(U)$ for all U. It turns out flasque sheaves also satisfy these conditions.