LECTURE 39
MATH 256B

LECTURE: PROFESSOR MARK HAIMAN
NOTES: JACKSON VAN DYKE

Today we would like to get to the point where we can actually compute sheaf cohomology.

1. Čech complexes

Let X be a ringed space, and let M be a sheaf of $\mathcal{O}_X$-modules. Let $\mathcal{U} = \{U_\alpha\}$ be a collection of open sets. Write $j_\alpha : U_\alpha \hookrightarrow X$. From this we will construct a complex. First we have the following:

$$0 \longrightarrow M \longrightarrow \prod_\alpha (j_\alpha)_* \left(M|_{U_\alpha}\right) \longrightarrow \prod_{\alpha,\beta} (j_{\alpha\beta})_* \left(M|_{U_\alpha \cap U_\beta}\right) \longrightarrow \cdots \ .$$

The nth term will look like

$$\prod_{|I|=n} (j_I)_* \left(M|_{U_I}\right)$$

and the map from the previous term is some sort of signed sum. This is called the Čech complex of a sheaf with respect to a covering.

Pick some particular index α, and then write

$$\mathcal{U}' = \{U_\beta\}_{\beta \neq \alpha} \qquad \mathcal{U}'' = (U_\beta \cap U_\alpha)_{\beta \neq \alpha} \ .$$

Then the complex $C^\bullet(M, \mathcal{U})$ is the mapping cone of

$$j : C^\bullet(M, \mathcal{U}') \to C^\bullet(M, \mathcal{U}'') \ .$$

Now restrict the complex $C^\bullet(M, \mathcal{U})|_{U_\alpha}$. This is acyclic, but then we can do this for any U_α. So as a complex of sheaves, $C^\bullet(M, \mathcal{U})$ is acyclic on $\cup_\alpha U\alpha$. Now we want to use this sort of thing to calculate the right derived functors $R\Gamma$ of the global sections functor.

2. Sheaf cohomology

Let $\mathcal{B}$ be a base of the topology on C with finite intersections. Then we say a sheaf M is $\mathcal{B}$-acyclic if

$$\Gamma_U C^\bullet(M, \mathcal{U})$$

is acyclic for all $\mathcal{U}$ which are coverings in $\mathcal{B}$ by $U \in \mathcal{B}$.

Consider an exact sequence of sheaves

$$0 \to A \to M \to N \to 0$$

where M is $\mathcal{B}$ acyclic. Then we want to see that for $U \in \mathcal{B}$ the following sequence is still exact:

$$0 \to A(U) \to M(U) \to N(U) \to 0 \ .$$

Date: May 3, 2019.

For $s \in N(\mathcal{U})$ there is some $t_0 \in M(U_\alpha)$ such that $t_0 \mapsto s|_{U_\alpha}$. Then we can consider

$$t_\alpha - t_\beta \in \Gamma_U C^2(M, \mathcal{U})$$

on $U_\alpha \cap U_\beta$. This is the boundary of $t_\alpha \in \Gamma_U(M, \mathcal{U})$. But this means the differential of this is 0. In fact, this is even in a smaller complex

$$(t_\alpha - t_\beta) \in \Gamma_U C^2(A, \mathcal{U})$$

and it is still a cycle. Then this comes from some $h_\alpha \in \Gamma_U C^1(A, \mathcal{U})$ under the map

$$\Gamma_U C^1(A, \mathcal{U}) \to \Gamma_U C^2(A, \mathcal{U}) \ .$$

These are somehow corrections. Define

$$t'_\alpha = t_\alpha - h_\alpha$$

and then

$$t'_\alpha - t'_\beta|_{U_\alpha \cap U_\beta} = 0$$

so these are somehow compatible.

The other thing we want to know is that if the first two are acyclic then so is the third. This turns out to follow from the above discussion.

The upshot is that $\mathcal{B}$-acyclic sheaves are acyclic for the functor Γ_U for any $U \in \mathcal{B}$.

2.1. **The fundamental theorem.**

Theorem 1. *If $\tilde{M}$ is a quasi-coherent sheaf on an affine scheme $X = \operatorname{Spec} R$, then*

$$R\Gamma\left(\tilde{M}\right) = \Gamma\left(\tilde{M}\right) \ ,$$

i.e. M is acyclic for Γ.

Remark 1. One might think this is obvious since Γ is exact on quasi-coherent sheaves. But the way to think of Γ is as a functor on all sheaves. Then we take the derived functor and this is true on quasi-coherent sheaves within this larger category. Somehow the category $\mathbf{QCoh}(X)$ is not such a good category since it doesnt have enough injectives and other issues. The proof is still somehow trivial.

Proof. Take $\mathcal{B} = \{X_f\}$. Then we show this is true for Γ_{X_f}. We already know if $\tilde{M}$ is finitely $\mathcal{B}$-acyclic then we are done. Take

$$X_f = \bigcup X_{f_i} \ .$$

All of the products in the Čech complex are finite so they're direct sums, and direct sums of qcoh sheaves are qcoh, and then we are done by exactness of Γ on qcoh sheaves. □

Corollary 1. *All products of quasi-coherent sheaves on $X = \operatorname{Spec} R$ are acyclic.*

This means even for an infinite covering, as long as the js are quasi-compact and separated morphisms then we can use a Čech complex as above to compute the sheaf cohomology of a quasi-coherent sheaf.

If X has the property that intersections of affines are affine (e.g. separated) then $\mathcal{B}$ can just be all affines. So even if X is not affine then this complex will still be acyclic on any of these affines.

3. An example

We will compute $H^i(\mathbb{P}^n_k, \mathcal{O}(d))$. We will do this over k, but every time we say vector space we could also say free-module and it would all still work.

Recall $\mathbb{P}^n_k = \operatorname{Proj} k[x_1, \ldots, x_{n+1}]$. Consider the following complex:

$$0 \longrightarrow k[\underline{x}] \longrightarrow \bigoplus_i k[\underline{x}]_{x_i} \longrightarrow \bigoplus k[\underline{x}]_{x_i,x_j} \longrightarrow \cdots \longrightarrow \bigoplus k[x]_{x_1,\ldots,x_{n+1}} \quad . \tag{1}$$

This complex is a complex of graded R-modules, and the sheaf associated to this on projective space is the Čech complex of $\mathcal{O}(d)$ with respect to the standard covering by affines.

We can basically see that this whole complex is in fact the tensor over k of

$$0 \to k[x] \to k\left[x, x^{-1}\right] \to 0 \ .$$

In this case the cohomology is:

$$H^0 = 0 \qquad\qquad H^1 = x^{-1}k\left[x^{-1}\right] \ .$$

The complex (1) is exact at every step except the last. But if we add the term

$$\ldots \to (x_1 \ldots x_{n+1})^{-1} k\left[x_1^{-1}, \ldots, x_{n+1}^{-1}\right] \to 0$$

then it is actually exact.

Now remove the first and last nontrivial term of (1). If we d-shift and take the degree 0 part of this then this exactly represents $R\Gamma\mathcal{O}(d)$ on $\mathbb{P}^n$. In the middle it has no cohomology, and then the cohomology at the ends is just the two end terms, i.e.

$$H^i(\mathbb{P}^n, \mathcal{O}(d)) = \begin{cases} k[\underline{x}]_{(d)} & i = 0 \\ 0 & 0 < i < n \\ (x_1 \ldots x_{n+1})^{-1} k\left[x_1^{-1}, \ldots, x_{n+1}^{-1}\right] & i = n \end{cases} \ .$$

Let's say we want to compute the cohomology of any coherent sheaf on projective space. We know this is associated to a f.g. graded module. Now we can find a f.g. graded free resolution of this module. Each term is a direct sum of degree shifts of the polynomial ring. I.e. they look like

$$\ldots \to \bigoplus \mathcal{O}(-d_j) \bigoplus \mathcal{O}(-d_i) \to \tilde{M} \ .$$

Then in principle we can compute the cohomology of M from this. We can in fact do these computations for any projective variety since we can just map it into projective space and make the computation.

In any case, just the existence of this means that if we think of M has an iterated mapping cone, then every complex like this will always have finite-dimensional cohomology in every degree.

If we tensor this entire complex with $\mathcal{O}(n)$ for large n, then eventually all of these negative degrees will become positive, and nobody has higher cohomology, only zero cohomology. I.e. we have resolved $\tilde{M}$ by acyclic objects. Then the conclusion turns out to be that

$$H^i(M \otimes \mathcal{O}(n)) = 0$$

for n sufficiently large. This is Serre's vanishing theorem.