LECTURE 4
MATH 256A

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1. Weighted projective space

Today we will consider the polynomial ring $R = k[x,y,z]$ and give each variable a different degree. In particular, we set $\deg x = \deg y = 1$, and $\deg z = 2$. Then we want to consider $\operatorname{Proj} R$. $\operatorname{Spec} R = \mathbb{A}^3$ as usual the action $k^\times \circlearrowright \mathbb{A}^3$ given by

$$t \cdot (x,y,z) = \left(tx, ty, t^2 z\right) \ .$$

Then the $k^\times$ orbits are curves which look like $\left(tx, ty, t^2 z\right)$. As usual, the points of $\operatorname{Proj} R$ are the irreducible $\mathbb{G}_m$ invariant (non-fixed) closed subsets. This is what is called weighted projective space. Our goal is to see what this looks like.

Remark 1. If we tried to do this with two variables it would just be the projective line.

We can do the usual thing where we cover this with affines:

$$\begin{aligned} X_x &= \operatorname{Spec} k\left[x^{\pm 1}, y, z\right]_0 \simeq \operatorname{Spec} k\left[x^{\pm 1}, y, z\right] / (x-1) \\ &= \operatorname{Spec} k\left[y/x, z/x^2\right] = k\left[y', z'\right] = \mathbb{A}^2 \ , \end{aligned}$$

and of course the same is true for $X_y = \mathbb{A}^2$. When we try to do the same for z we get

$$X_z = \operatorname{Spec} k\left[x, y, z^{\pm 1}\right]_0 \ ,$$

but we can't proceed as we did with the other coordinates. One way to think about this is to use the fact that we saw last time:

$$\operatorname{Proj} R \cong \operatorname{Proj}\left(R_{n\mathbb{Z}}\right) \ .$$

Remark 2. Though these are the same as schemes, they are actually different in some sense, because the distinguished invertible sheaf on the right is the nth tensor power of the distinguished invertible sheaf on the left.

In this case we have

$$R_{2\mathbb{Z}} = k\left[x^2, xy, y^2, z\right] = k[r,s,t,z] / \left(s^2 - rt\right)$$

which means

$$X = \operatorname{Proj} R = \operatorname{Proj} k[r,s,t,z] / \left(s^2 - rt\right) \cong V\left(s^2 - rt\right) \subseteq \mathbb{P}^3_k \ .$$

X_z is the same in this picture, and in particular

$$X_z = \operatorname{Spec} k[r,s,t] / \left(s^2 - rt\right)$$

Date: January 30, 2019.

where these variables are really $/z$ relative to what they were above. The picture here is that we have some sort of three space with coordinates r, s, and t. Then there is a $\mathbb{P}^2$ at ∞, which corresponds to $z = 0$ then we have an affine cone which we can take the projective completion of to get a cone in $\mathbb{P}^2$.

Remark 3. We might wonder what happens when we have countably infinitely many variables with different weights. First of all, whatever $\mathbb{A}^\infty$ and $\mathbb{P}^\infty$ are[1] we can write

$$\operatorname{Spec}\left(k\left[x_1, x_2, \cdots\right]\right) = \mathbb{A}_k^\infty \qquad \operatorname{Proj}\left(k\left[x_1, x_2, \cdots\right]\right) = \mathbb{P}_k^\infty \ .$$

The first is quasi-compact, and the second isn't. When we allow different weights, but force them to be bounded in somehow generalizes in the way we would expect, but if the degrees become unbounded more things begin to break.

2. Quotienting by group schemes

Now we want to understand the sense in which $\operatorname{Proj} R$ is the 'quotient' of $\operatorname{Spec}(R) \setminus V(R_+)$ by the $\mathbb{G}_m$ action. First we will consider the strongest possible notion of a quotient.

Let Z be a scheme, and G a group scheme acting on Z. Then the obvious idea would be to define:

$$\underline{\underline{Z/G}}(T) = \underline{\underline{Z}}(T) / \underline{\underline{G}}(T) \ .$$

But this hardly ever works. It does however happen to work in the following setting. If $Z = A \times_S G$ (as schemes over S) where G acts trivially on A and on G just as usual, then our intuition tells us that $Z/G = A$. But the problem is that this is basically the only way it can happen. To see why, consider the alleged map $Z \to Z/G = T$. Then T points of Z are morphisms $T \to Z$, and T points of Z/G are maps $T \to T$. One such point is the identity map. The diagram here is:

$$\begin{tikzcd} & Z \arrow[d] \\ T \arrow[ur, "\sigma"] \arrow[r, "\mathrm{id}"] & Z/G \end{tikzcd} \ .$$

Once we have a section σ, if G acts freely, we have an isomorphism of $Z \simeq (Z/G) \times G$ so this is the only way. But most group actions that are free aren't trivial like this. The general notion of such a thing is that we have a principal G bundle $Z \to Z/G$. Freely here just means the fibers are faithful.

Example 1. Take $\mathbb{C}^\times \to \mathbb{C}^\times$ to map $z \mapsto z^2$. This is a morphism of algebraic varieties, and group homomorphism of algebraic groups with nontrivial kernel $G = \{\pm 1\}$ so every fiber has a free action of this kernel. This doesn't have a section even in the analytic topology. I.e. there is no global square root, but we can of course take branch cuts. So this is a non-principal bundle for this G, but it is locally a principal G bundle.

It turns out the right notion is somehow a 'local' version of this functorial quotient. This will turn out to work if R_+ is generated by degree 1 elements. Then there is a weaker version which works for every R. We will continue this discussion next lecture.

[1] Somehow the obvious thing. . .