LECTURE 5 MATH 256A

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1. Quotients

1.1. Classical picture. We want to finish discussing the sense in which we can think of $\operatorname{Proj} R$ as a quotient of $\operatorname{Spec} R \setminus V(R_+)$ by the \mathbb{G}_m action. Before we get bogged down with abstraction, let's remember the classical story. Let $A = \mathcal{O}(Y)$, then $\operatorname{Proj}(A[x_0, \cdots, x_n]/I)$ is a projective variety. In this case $\operatorname{Spec} R$ is $\mathbb{A}^{n+1} \times Y$ and the k^{\times} action is just by scaling. So we remove the fixed locus, a copy of Y, and quotient out by k^{\times} to get $Y \times \mathbb{P}^n_k$.

1.2. **Generality.** First we need some sort of canonical morphism : π : Spec $(R) \setminus V(R_+) \to \operatorname{Proj}(R)$. The way we constructed Proj to begin with was somehow the wrong way around. To see this as a quotient, we want to view it as sending points to their G-orbits. So let's see we actually have such a map. Let $Z = \operatorname{Spec} R$ and $X = \operatorname{Proj} R$. As usual we have that X is covered by the $X_f = \operatorname{Spec}(R_f)_0$, and Z is covered by the $Z_f = \operatorname{Spec}(R_f)$ so locally we have maps $Z_f \to X_f$, and the global map is pasted together like this. Note that this does indeed map fixed points to fixed points, and non-fixed points to their orbits, so this is a quotient in a topological sense.

There are many sorts of quotients. One is the sort of functorial quotient:

$$\underline{Y}(T) = \underline{X}(T) / \underline{G}(T)$$
.

But this is too strong because if this is true, then $X = Y \times G$ to begin with. We also have a 'locally-functorial' notion of a quotient, e.g. X could be a principal G bundle over Y in the Zariski topology. Finally there is a weaker notion, which is what is called a "coarse" quotient. So consider an action $G \odot X$, and a G-equivariant map $\pi: X \to Y$ such that $G \odot Y$ trivially. Then the property we want is just that it is universal among such things. I.e. for every G-equivariant map $\varphi: X \to Z$, where Z has trivial G action, we have the following:

$$\begin{array}{c}
Y \\
\downarrow \exists ! \\
X \xrightarrow{\varphi} Z
\end{array}$$

This notion also seems functorial, but it turns out it is much weaker. We want to see the functorial quotient as a special case of the coarse quotient. Note that this will imply that the locally functorial quotient is also a special case. So suppose we do have a functorial quotient. Then given $\varphi: X \to Z$, consider the induced map $\underline{X}(T) \to \underline{Z}(T)$ for any scheme T, which is equivariant with respect

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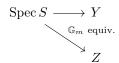
to $\underline{\underline{G}}(T) \odot \underline{\underline{X}}(T)$. Then for $\underline{\underline{Y}}(T) = \underline{\underline{X}}(T)/\underline{\underline{G}}(T)$ the functorial quotient, it fits in the diagram:

$$\underbrace{\underline{\underline{Y}}}_{T}(T)$$

$$\underline{\underline{X}}(T) \xrightarrow{\varphi} \underline{\underline{Z}}(T)$$

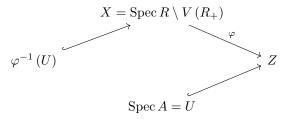
and by Yoneda, there exists a unique such map $Y \to Z$.

1.3. **Specifics.** First let's see that the map $\operatorname{Spec} R \setminus V(R_+) \to \operatorname{Proj} R$ is a weak quotient. The action $\mathbb{G}_m \subset \operatorname{Spec} S$ gives us a trivial \mathbb{Z} grading in S, meaning $S = S_0$. For the moment let $Z = \operatorname{Spec} A$ be an affine scheme. We are supposed to think of Z as having a trivial \mathbb{G}_m action



So this is a weak weak quotient.

Now consider



and notice we can cover $\varphi^{-1}(U)$ with X_f s for $f \in R_d$ for d > 0. Then the maps between affines correspond to maps $A \to (R_f)_0$. But this says exactly that for $Y = \operatorname{Proj} R$ and $Y_f = \operatorname{Spec}(R_f)_0$ we have that the morphism $X_f \to U$ factors through Y_f , which gives us a unique morphism $\operatorname{Proj} R \to Z$ making the diagram commute:

$$Y = \operatorname{Proj} R$$

$$\downarrow \exists !$$

$$X = \operatorname{Spec} R \setminus V(R_{+}) \xrightarrow{\varphi} Z$$

Example 1. Take the affine line Spec k[x]. The fixed point is 0, so we get $k^{\times}/k^{\times} = \text{pt} \simeq \text{Spec } k$. Indeed $X_x = \text{Spec } k[x^{\pm 1}]_0 = \text{Spec } k$ covers it. The issue with the weak quotient if we don't remove the origin, is that there are two orbits, but we just get one point.

Example 2. Now consider k[x,y]. We have $k[x,y]_0 = k$, so the weak quotient is a point when we don't remove the origin. The idea is that the origin is in the closure of every orbit. But in a sensible quotient, we need to distinguish the orbits, so we remove the fixed locus.

Suppose that R_1 generates R_+ .

Exercise 1. Prove this is equivalent to the condition that R_0 and R_1 generate R as a ring.

Actually we just need $R_+ \subseteq \sqrt{R}$. I.e. the Y_f , for deg f = 1, cover Spec $R \setminus V(R_+)$ and the X_f cover Proj R. As usual let $X_f = \operatorname{Spec} R_f$ and $Y_f = \operatorname{Spec} (R_f)_0$. Now we have

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$$\begin{array}{c} (R_f)_n \\ f^n \uparrow \downarrow f^{-n} \\ (R_f)_0 \end{array}$$

and then $\operatorname{Spec} R_f = \operatorname{Spec}(R_f)_0 \times_{\operatorname{Spec} \mathbb{Z}} \mathbb{G}_{m,\mathbb{Z}}$ is a strong quotient, and $X = \operatorname{Spec} R \setminus V(R_+) \to Y = \operatorname{Spec} R$ is a principal \mathbb{G}_m bundle.

2. Preview of Next time

On affine schemes $X = \operatorname{Spec} R$, if we have an R-module M, we get a sheaf \tilde{M} of \mathcal{O}_X modules. In particular we have the stalks $\tilde{M}_{\mathfrak{p}} = M_{\mathfrak{p}}$. Then we had a theorem that we also have $\tilde{M}(X_f) = M_f$. As a special case, $\Gamma\left(\tilde{M}\right) = M$. In other words we have an adjunction

$$R$$
-Mod $\Gamma \cap \tilde{\Gamma}$: \mathcal{O}_X -Mod

In particular $\tilde{\cdot}$ is left adjoint to Γ . So this is an equivalence of categories between $R\text{-}\mathbf{Mod}$ and its image in $\mathcal{O}_X\text{-}\mathbf{Mod}$. The image consists of quasi-coherent $\mathcal{O}_X\text{-}$ modules.

As it turns out, we can do the same for Proj. In particular, we take $X = \operatorname{Proj} R$, and then we map M a graded R-module to \tilde{M} an \mathcal{O}_X module. It works basically the same, except now

$$\tilde{M}(X_f) = (\tilde{M_f})_0$$
.