LECTURE 5
MATH 256A

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1. Quotients

1.1. **Classical picture.** We want to finish discussing the sense in which we can think of $\operatorname{Proj} R$ as a quotient of $\operatorname{Spec} R \setminus V(R_+)$ by the $\mathbb{G}_m$ action. Before we get bogged down with abstraction, let's remember the classical story. Let $A = \mathcal{O}(Y)$, then $\operatorname{Proj}(A[x_0, \cdots, x_n]/I)$ is a projective variety. In this case $\operatorname{Spec} R$ is $\mathbb{A}^{n+1} \times Y$ and the $k^\times$ action is just by scaling. So we remove the fixed locus, a copy of Y, and quotient out by $k^\times$ to get $Y \times \mathbb{P}^n_k$.

1.2. **Generality.** First we need some sort of canonical morphism : $\pi : \operatorname{Spec}(R) \setminus V(R_+) \to \operatorname{Proj}(R)$. The way we constructed Proj to begin with was somehow the wrong way around. To see this as a quotient, we want to view it as sending points to their G-orbits. So let's see we actually have such a map. Let $Z = \operatorname{Spec} R$ and $X = \operatorname{Proj} R$. As usual we have that X is covered by the $X_f = \operatorname{Spec}(R_f)_0$, and Z is covered by the $Z_f = \operatorname{Spec}(R_f)$ so locally we have maps $Z_f \to X_f$, and the global map is pasted together like this. Note that this does indeed map fixed points to fixed points, and non-fixed points to their orbits, so this is a quotient in a topological sense.

There are many sorts of quotients. One is the sort of functorial quotient:

$$\underline{\underline{Y}}(T) = \underline{\underline{X}}(T) / \underline{\underline{G}}(T) \ .$$

But this is too strong because if this is true, then $X = Y \times G$ to begin with. We also have a 'locally-functorial' notion of a quotient, e.g. X could be a principal G bundle over Y in the Zariski topology. Finally there is a weaker notion, which is what is called a "coarse" quotient. So consider an action $G \circlearrowright X$, and a G-equivariant map $\pi : X \to Y$ such that $G \circlearrowright Y$ trivially. Then the property we want is just that it is universal among such things. I.e. for every G-equivariant map $\varphi : X \to Z$, where Z has trivial G action, we have the following:

$$\begin{array}{ccc} & & Y \\ & \overset{\pi}{\nearrow} & \downarrow {\scriptstyle \exists !} \\ X & \xrightarrow{\varphi} & Z \end{array} \ .$$

This notion also seems functorial, but it turns out it is much weaker. We want to see the functorial quotient as a special case of the coarse quotient. Note that this will imply that the locally functorial quotient is also a special case. So suppose we do have a functorial quotient. Then given $\varphi : X \to Z$, consider the induced map $\underline{\underline{X}}(T) \to \underline{\underline{Z}}(T)$ for any scheme T, which is equivariant with respect

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to $\underline{\underline{G}}(T) \circlearrowright \underline{\underline{X}}(T)$. Then for $\underline{\underline{Y}}(T) = \underline{\underline{X}}(T) / \underline{\underline{G}}(T)$ the functorial quotient, it fits in the diagram:

$$\begin{array}{ccc} & & \underline{\underline{Y}}(T) \\ & \overset{\pi}{\nearrow} & \downarrow \\ \underline{\underline{X}}(T) & \xrightarrow{\varphi} & \underline{\underline{Z}}(T) \end{array}$$

and by Yoneda, there exists a unique such map $Y \to Z$.

1.3. **Specifics.** First let's see that the map $\operatorname{Spec} R \setminus V(R_+) \to \operatorname{Proj} R$ is a weak quotient. The action $\mathbb{G}_m \circlearrowright \operatorname{Spec} S$ gives us a trivial $\mathbb{Z}$ grading in S, meaning $S = S_0$. For the moment let $Z = \operatorname{Spec} A$ be an affine scheme. We are supposed to think of Z as having a trivial $\mathbb{G}_m$ action

$$\begin{array}{ccc} \operatorname{Spec} S & \longrightarrow & Y \\ & \underset{\mathbb{G}_m \text{ equiv.}}{\searrow} & \\ & & Z \end{array}$$

So this is a weak weak quotient.

Now consider

$$\begin{array}{ccccc} & & X = \operatorname{Spec} R \setminus V(R_+) & & \\ & \nearrow & & \overset{\varphi}{\searrow} & \\ \varphi^{-1}(U) & & & & Z \\ & & & \nearrow & \\ & & \operatorname{Spec} A = U & & \end{array}$$

and notice we can cover $\varphi^{-1}(U)$ with X_fs for $f \in R_d$ for $d > 0$. Then the maps between affines correspond to maps $A \to (R_f)_0$. But this says exactly that for $Y = \operatorname{Proj} R$ and $Y_f = \operatorname{Spec}(R_f)_0$ we have that the morphism $X_f \to U$ factors through Y_f, which gives us a unique morphism $\operatorname{Proj} R \to Z$ making the diagram commute:

$$\begin{array}{ccc} & & Y = \operatorname{Proj} R \\ & \nearrow & \downarrow \exists ! \\ X = \operatorname{Spec} R \setminus V(R_+) & \xrightarrow{\varphi} & Z \end{array} .$$

Example 1. Take the affine line $\operatorname{Spec} k[x]$. The fixed point is 0, so we get $k^\times / k^\times = \mathrm{pt} \simeq \operatorname{Spec} k$. Indeed $X_x = \operatorname{Spec} k\left[x^{\pm 1}\right]_0 = \operatorname{Spec} k$ covers it. The issue with the weak quotient if we don't remove the origin, is that there are two orbits, but we just get one point.

Example 2. Now consider $k[x, y]$. We have $k[x, y]_0 = k$, so the weak quotient is a point when we don't remove the origin. The idea is that the origin is in the closure of every orbit. But in a sensible quotient, we need to distinguish the orbits, so we remove the fixed locus.

Suppose that R_1 generates R_+.

Exercise 1. Prove this is equivalent to the condition that R_0 and R_1 generate R as a ring.

Actually we just need $R_+ \subseteq \sqrt{R}$. I.e. the Y_f, for $\deg f = 1$, cover $\operatorname{Spec} R \backslash V(R_+)$ and the X_f cover $\operatorname{Proj} R$. As usual let $X_f = \operatorname{Spec} R_f$ and $Y_f = \operatorname{Spec}(R_f)_0$. Now we have

$$\begin{array}{c} (R_f)_n \\ f^n \Big\uparrow \Big\downarrow f^{-n} \\ (R_f)_0 \end{array}$$

and then $\operatorname{Spec} R_f = \operatorname{Spec}(R_f)_0 \times_{\operatorname{Spec} \mathbb{Z}} \mathbb{G}_{m,\mathbb{Z}}$ is a strong quotient, and $X = \operatorname{Spec} R \backslash V(R_+) \to Y = \operatorname{Spec} R$ is a principal $\mathbb{G}_m$ bundle.

2. Preview of next time

On affine schemes $X = \operatorname{Spec} R$, if we have an R-module M, we get a sheaf $\tilde{M}$ of $\mathcal{O}_X$ modules. In particular we have the stalks $\tilde{M}_{\mathfrak{p}} = M_{\mathfrak{p}}$. Then we had a theorem that we also have $\tilde{M}(X_f) = M_f$. As a special case, $\Gamma\left(\tilde{M}\right) = M$. In other words we have an adjunction

$$\begin{array}{c} R\text{-}\mathbf{Mod} \\ \Gamma \Big\uparrow \Big\downarrow \tilde{\cdot} \\ \mathcal{O}_X\text{-}\mathbf{Mod} \end{array} .$$

In particular $\tilde{\cdot}$ is left adjoint to Γ. So this is an equivalence of categories between R-**Mod** and its image in $\mathcal{O}_X$-**Mod**. The image consists of quasi-coherent $\mathcal{O}_X$-modules.

As it turns out, we can do the same for Proj. In particular, we take $X = \operatorname{Proj} R$, and then we map M a graded R-module to $\tilde{M}$ an $\mathcal{O}_X$ module. It works basically the same, except now

$$\tilde{M}(X_f) = \left(\tilde{M_f}\right)_0 .$$