LECTURE 6
MATH 256A

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Today we will talk more about sheaf theory on $\operatorname{Proj} R$. We will need some missing pieces of theory on affine schemes. In particular we want to better understand the reason that there are sheaves of $\mathcal{O}_X$-modules which don't come from R-modules.

1. Sheaves and global sections

Let $X = \operatorname{Proj} R$. Then consider a graded R-module M. From this we get a sheaf $\tilde{M}$ of $\mathcal{O}_X$ modules which can be described in 'coordinates' $X_f = \operatorname{Spec} R\left[f^{-1}\right]_0$ to be

$$\tilde{M}\Big|_{X_F} = M\left[f^{-1}\right]_0^{\tilde{}} \ .$$

So this is sort of the same as the affine case so far. But just looking at examples, we will find that it is sort of more subtle.

Example 1. Consider $M = R$ with the obvious grading. Then $R^{\tilde{}} = \mathcal{O}_X$.

Example 2. We can always just change the grading by adding something to every degree. I.e. $M[d]$ is the same module, except now the degree n part is $M[d]_n = M_{d+n}$. In particular, we can consider $R[d]^{\tilde{}}$ as a module. Locally $\mathcal{O}_X(d) = R[d]^{\tilde{}}$ looks like

$$R[d]^{\tilde{}}\Big|_{X_f} = R\left[f^{-1}\right]_d^{\tilde{}}$$

where the $\tilde{}$ on the right is for Spec and on the left it is for Proj. If the X_f for $f \in R_1$ cover X, i.e. $R_+ \subset \sqrt{(R_1)}$, then $\mathcal{O}_X(d)$ is locally isomorphic to $\mathcal{O}_X$:

$$\mathcal{O}_X(d)|_{X_f} \cong \mathcal{O}_X|_{X_f}$$

but these are not necessarily isomorphic globally, since they might not agree on the overlaps. Usually these are not isomorphic, even if $V(R_1) = \emptyset$, i.e. open sets defined by degree one elements cover the entire thing.

Remark 1. Any element $a \in M_0$ also gives us an element $a/1 \in M\left[f^{-1}\right]_0$ so we get a global section $\tilde{a} \in \tilde{M}(X)$. For $a \in M_d$, we get $\tilde{a} \in \widetilde{M[d]}(X)$ and we have a map $R_d \to \mathcal{O}_X(d)(X)$. For notation we will write $\mathcal{O}_X(d)(X) = \Gamma(X, \mathcal{O}_X(d))$.

2. Recovering a graded module using global sections

The following is a well-behaved example:

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Example 3. Consider the classical case:

$$X = \mathbb{P}^n_k = \operatorname{Proj} k\left[x_0, \cdots, x_n\right] = \operatorname{Proj} R\ .$$

So R_d is the space of homogeneous polynomials of degree d. A priori functions of degree d don't give functions on X. Ratios of them do, but these are somehow not defined on all lines in $\mathbb{A}^{n+1}$ since the denominator might vanish. We do have a map $R_d \to \Gamma\left(\mathbb{P}^n_k, \mathcal{O}\left(d\right)\right)$. This is covered by the

$$X_i \coloneqq X_{x_i} = \operatorname{Spec} k\left[\frac{x_0}{x_i}, \cdots, \left(\frac{x_i}{x_i}\right), \cdots, \frac{x_n}{x_i}\right]$$

where we have omitted x_i/x_i. So x_i^d gives us a global section of $\mathcal{O}_X\left(d\right)$ which is a generator on X_i. I.e. we have an isomorphism $\mathcal{O}_{X_i} \xrightarrow{\sim} \mathcal{O}_{X_i}\left(d\right)$ which sends $1 \mapsto x_i^d$.

This means that we have an explicit description

$$\mathcal{O}_X\left(d\right)\left(X_i\right) = \left\{x_i^d g_i\left(\frac{x_0}{x_i}, \cdots, \frac{x_n}{x_i}\right)\right\}\ .$$

Then for $f \in R_d$ the map is:

$$f\left(x_0, \cdots, x_n\right) \mapsto x_i^d f\left(\frac{x_0}{x_i}, \cdots, 1, \cdots, \frac{x_n}{x_i}\right)\ .$$

But if f is homogeneous and we set one variable equal to 1, we haven't lost any information, since f is homogeneous. This means $R_d \hookrightarrow \Gamma\left(X, \mathcal{O}_X\left(d\right)\right)$.

Now what about the global functions. This means we have to specify it on each X_i and that they are compatible. I.e. for whatever generators σ_i, we have

$$\sigma_i g_i\left(\frac{x_0}{x_i}, \cdots, \frac{x_n}{x_i}\right) = \sigma_j g_j\left(\frac{x_0}{x_j}, \cdots, \frac{x_n}{x_i}\right)$$

where for us we just have $\sigma_i = x^i$, so this equality takes place in $k\left[\underline{x}, x_i^{-1}, x_j^{-1}\right]$. But these only have x_i (resp. x_j) in their denominators, so we can clear denominators so they all agree with one actual polynomial in degree d. But this is the polynomial f.

Note for $d < 0$ there are no global sections at all. But what will happen is that there will be some sort of higher sheaf cohomology.

Note that a special case of this is $\mathcal{O}_{\mathbb{P}^n}\left(\mathbb{P}^n\right) = k$.

This example was supposed to mislead us into thinking that we get the graded module back degree by degree by taking global sections. In fact this even worked for negative degrees. So now we will do an example which will make us less comfortable.

Example 4. Consider another (just as classical) example, the two-point scheme $X = \{0, \infty\} = V\left(xy\right) \subseteq \mathbb{P}^1$. Note that $X = \operatorname{Proj} k\left[x, y\right]/\left(xy\right)$. Professor Haiman isn't pulling the wool over our eyes by writing this because of the following. We know Proj somehow only cares about large degrees, since we can do this thinning procedure, i.e. we could take $V\left(x^2y, xy^2\right)$ as well. But this ideal is sort of not saturated, i.e. we have an issue with the degree 2 part. But the ideal $\left(xy\right)$ is saturated.

The trouble with this example is that $\mathcal{O}_X\left(d\right) \cong \mathcal{O}_X$ sort of trivially since X is a discrete space. We immediately have that $\Gamma\left(X, \mathcal{O}_X\left(d\right)\right) \cong k^2$ for every d. We of course have a map

$$k\left[x, y\right]/\left(xy\right) \to \Gamma\left(X, \mathcal{O}_X\left(d\right)\right) \cong k^2\ .$$

In degree 0 we have $R_0 = k$, for $d < 0$, $R_d = 0$, and for $d > 0$ we have $R_d = k^2$. So in degree 0 it is injective but not surjective, for negative degrees it is nothing, and it is an isomorphism in positive degrees. So we can't actually recover the module using global sections.

In this example we did however see that it works in high degrees. As it turns out, under some quasi-compactness assumptions and some other mild assumptions, we will be able to do this recovery.

Let's think about $X \hookrightarrow Y = \mathbb{P}^2$. In particular, this corresponds to a map $R \to R/I$. Then we can consider $\widetilde{R/I} = i_*\mathcal{O}_X$ on $\operatorname{Proj} R$, then we have that this is a quotient sheaf $\mathcal{O}_X \twoheadrightarrow i_*\mathcal{O}_X$. So we have

$$\mathcal{O}_{\mathbb{P}^1}(d) \twoheadrightarrow M$$

but $M = i_*\mathcal{O}_X(d)$. Now we might expect taking global sections to preserve the surjectivity, since it did for Spec. But taking global sections on the left gives us degree d polynomials, and on the right we get the points 0 and ∞. Explicitly the map is evaluation. But this is only surjective in high degrees. I.e. we have discovered that Γ is not exact on Proj. This is why these are the first examples with interesting sheaf cohomology.