LECTURE 7
MATH 256B

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We will cover some aspects of sheaf theory on affine schemes which we didn't cover last semester.

1. Affine schemes

Let $X = \operatorname{Spec} R$, and M be an R-module. We then created a sheaf $\tilde{M}$ of $\mathcal{O}_X$-modules. Then we had a theorem

Theorem 1. *The global sections are* $\tilde{M}(X) = M$.

We also have that

$$\tilde{M}(X_f) = M_f$$

where $X_f = \operatorname{Spec} R\left[f^{-1}\right]$. We also saw that this was an exact functor. Then we have the obvious functor Γ which just takes global sections. Then this theorem is saying that Γ is left inverse to $\tilde{\cdot}$. The opposite however isn't true. We will give some examples of this. Note that even more is true: Γ is actually left adjoint to $\tilde{\cdot}$. I.e. for some $\mathcal{O}_X$-module $\mathcal{N}$, we have a canonical isomorphism

$$\operatorname{Hom}_R(M, \Gamma(\mathcal{N})) \simeq \operatorname{Hom}_{\mathcal{O}_X}\left(\tilde{M}, \mathcal{N}\right) .$$

This is true because of the following. Let $\sigma \in \operatorname{Hom}_{\mathcal{O}_X}\left(\tilde{M}, \mathcal{N}\right)$. Giving this is equivalent to giving this for each X_f:

$$M_f \to \mathcal{N}(X_f) .$$

These of course have to be given compatibly. And the universal property of localization tells us that this is the same as an R-module homomorphism $M \to \mathcal{N}(X_f)$ which comes from a morphism

$$M \to \mathcal{N}(X) .$$

Now we want to somehow intrinsically identify the image of the $\tilde{\cdot}$ functor. To answer this we need a definition which is somehow independent of talking about sheaves associated to modules:

Definition 1. For an $\mathcal{O}_X$-module $\mathcal{M}$, we call it *quasi-coherent* (qco) if X can be covered by open sets U such that $\mathcal{M}|_U$ has a presentation as:

$$\left.\mathcal{O}_X^{(J)}\right|_U \to \left.\mathcal{O}_X^{(I)}\right|_U \to \mathcal{M}|_U \to 0$$

where we allow the indexing set I and J to be infinite. I.e. we have a presentation of this as a cokernel of the first map.

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Remark 1. This definition makes sense for X any ringed space, but is basically only useful for schemes.

Let X be a scheme. If this happens for some open covering, we can cover each of the opens by affines, so WLOG we can take $U = \operatorname{Spec} R$ to cover X. Note that localization preserves direct sums, which means we actually have:

$$\tilde{R}^{(J)} \to \tilde{R}^{(I)}$$

which just comes from an R-module homomorphism with cokernel M:

$$R^{(J)} \to R^{(I)} \to M \to 0$$

but since $\tilde{\cdot}$ is exact, we have that the cokernel of the first map is $\tilde{M}$:

$$\tilde{R}^{(J)} \to \tilde{R}^{(I)} \to \tilde{M} \to 0$$

So $\tilde{M}$ is certainly quasi-coherent. Now if X is an affine scheme itself, then we claim that the quasi-coherent schemes are all of the form $\tilde{M}$.

Proposition 1. *If $X = \operatorname{Spec} R$ and $\mathcal{M}$ is quasi-coherent, then $\mathcal{M} = \tilde{M}$ where $M = \mathcal{M}(X)$.*

Proof. We want to show that $\mathcal{M}(X_f) = M_f$. Cover X with opens:

$$X = \bigcup_i X_{g_i}$$

so we have

$$\mathcal{M}|_{X_{g_i}} = \widetilde{M_i} \ .$$

But now if we consider the following intersections we have:

$$\mathcal{M}(X_{g_i} \cap X_f) = \mathcal{M}(X_{g_i})_f$$

and similarly

$$\mathcal{M}\left(X_{g_i} \cap X_{g_j} \cap X_f\right) = \mathcal{M}\left(X_{g_i} \cap X_{g_j}\right)_f \ .$$

Now the sections on all of X can be described as:

$$0 \to \mathcal{M}(X) \to \bigoplus_i \mathcal{M}(X_{g_i}) \to \bigoplus_{i,j} \mathcal{M}\left(X_{g_i} \cap X_{g_j}\right)$$

where the global sections are the kernel of this map because of the sheaf axiom. This works the same way on the X_fs:

$$0 \to \mathcal{M}(X_f) \to \bigoplus_i \mathcal{M}(X_{g_i} \cap X_f) \to \bigoplus_{i,j} \mathcal{M}\left(X_{g_i} \cap X_{g_j} \cap X_f\right) \ .$$

But now notice that from the above observations we have the two maps given by localization:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{M}(X) & \longrightarrow & \bigoplus_i \mathcal{M}(X_{g_i}) & \longrightarrow & \bigoplus_{i,j} \mathcal{M}\left(X_{g_i} \cap X_{g_j}\right) \\ & & & & \downarrow (-)_f & & \downarrow (-)_f \\ 0 & \longrightarrow & \mathcal{M}(X_f) & \longrightarrow & \bigoplus_i \mathcal{M}(X_{g_i} \cap X_f) & \longrightarrow & \bigoplus_{i,j} \mathcal{M}\left(X_{g_i} \cap X_{g_j} \cap X_f\right) \end{array} .$$

But since localization is an exact functor, we get a map on the kernels as well, i.e. we have:

$$(-)_f : \mathcal{M}(X) \to \mathcal{M}(X_f) = \mathcal{M}(X)_f \ .$$

□

One thing that $\mathcal{M}(X_f) = \mathcal{M}(X)_f$ is saying, is that if $\sigma \in \mathcal{M}(X)$ has $\sigma|_{X_f} = 0$ then this is equivalent to some power n killing it: $f^n\sigma = 0$. It is also saying that for every section $\tau \in \mathcal{M}(X_f)$ there exists some n such that $f^n\tau$ extends to some $\sigma \in \mathcal{M}(X)$.

2. Examples

Let's have some examples of sheaves of $\mathcal{O}_X$-modules (on affine schemes) that are not in the image of the localization functor. In other words, for $X = \operatorname{Spec} R$, we want to find some $\mathcal{O}_X$ module $\mathcal{M}$ which is not qco. There are three main types of examples of these.

Example 1. The first is easy but artificial such as $R = k[x]_{(x)}$ (or $R = \mathbb{Z}_{(p)}$). In this case, $\operatorname{Spec} R = \{Q = (0), P = (x)\}$ so $Q \to P$. The only nonempty open sets are $X = \overline{\{Q\}}$, and $\{Q\}$. To give a sheaf, we need to specify it on the open sets X and $\{Q\}$:

$$\mathcal{M}(X) = \mathcal{M}_P \qquad\qquad M(\{Q\}) = \mathcal{M}_Q$$

and give a restriction map $\mathcal{M}_P \to \mathcal{M}_Q$. We know $\mathcal{O}_P = Q$ and $\mathcal{O}_Q = k(x)$. So we need to give an R-module, a vector space over $k(x)$, and a homomorphism between them that is basically an R-module homomorphism. So take any R-module A, and then this maps to $k(x) \otimes_R A$ which maps $k(x) \otimes_R A \to B$ for whatever B we choose, but $\tilde{A}$ is A with a map to its localization $A \to k(x) \otimes_R A$ so if $B \neq k(x) \otimes_R A$ then it isn't qco.

Example 2. Now take $R = k[x]$, so $X = \operatorname{Spec} R = \mathbb{A}^1_k$. One of the points is the origin $0 = (x)$, and we have an open $U = \mathbb{A}^1_k \setminus \{0\} = \mathbb{G}_{m,k}$. In general, for an open subset $j : U \hookrightarrow X$ and a sheaf $\mathcal{A}$ on U, we can form a new sheaf by 'extending it by 0' to get $j_!\mathcal{A}$, which is called $\mathcal{A}$ shriek. It is defined as follows. First of all, for an open V we have

$$j_!\mathcal{A}(V) \subseteq \mathcal{A}(V \cap U) \ .$$

If $V \subseteq U$, we have that

$$\mathcal{A}(V) = \{\sigma \in \mathcal{A}(V \cap U) \mid \forall P \notin V \setminus U, \exists \text{ nbhd } P \in W \text{ s.t. } \sigma|_{X \cap U \cap W} = 0\}$$

so

$$j_!\mathcal{A}|_U = \mathcal{A}$$

and for $P \notin U$,

$$j_!\mathcal{A}_P = 0 \ .$$

So for $X = \mathbb{A}^1_k$ and U the affine line without the origin, the global sections are

$$\Gamma(X, j_!\mathcal{O}_U) = 0$$

since all the open sets are of the form U_f for some f. So this can't be the sheaf associated to an R-module, and $j_!\mathcal{O}_U$ cannot be qco.

Example 3. We know tensor products preserve direct sums but not infinite products. For the same reason $\tilde{\ }$ will not preserve infinite products. This gives us a third class of examples. Most of the time an infinite product of things of the form $\tilde{M}$ will not be qco.

Exercise 1. Give an explicit example of this.