LECTURE 8
MATH 256A

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Today will be a bit more of sheaf basics. We will discuss the functors on qco sheaves a bit more. Let $\varphi : X \to Y$. Then we have the direct image functor φ_*, and the φ^{-1} functor. This is fine on the level of sets, i.e. φ^{-1} is adjoint to φ_* in this case, but we have to promote φ^{-1} to some functor φ^* in order to be adjoint for modules, i.e. to get a proper sheaf of $\mathcal{O}_X$-modules from sheaves of $\mathcal{O}_Y$ modules.

1. Affine schemes

Let $\varphi : X = \operatorname{Spec} A \to Y = \operatorname{Spec} B$ be a map of affine schemes with corresponding ring homomorphism $\alpha : B \to A$. Then for $\mathcal{M} \in \mathbf{QCoh}(X)$ we want to understand what the correct notion of $\varphi^* \mathcal{M}$ is. Recall that a morphism of ringed spaces really consists of the data $\left(\varphi, \varphi^\flat, \varphi^\#\right)$ where

$$\varphi^\flat : \mathcal{O}_Y \to \varphi^* \mathcal{O}_X \qquad\qquad \varphi^\# : \varphi^{-1} \mathcal{O}_Y \to \mathcal{O}_X \ .$$

Then $\varphi_* \mathcal{M}$ is just sort of automatically a $\varphi_* \mathcal{O}_X$-module, but the map $\varphi^\flat$ also gives this the structure of an $\mathcal{O}_Y$ module.

The inverse image functor however isn't so nice. If we start with an $\mathcal{O}_Y$-module, then we can take $\varphi^{-1} \mathcal{N}$, which is naturally a $\varphi^{-1} \mathcal{O}_Y$-module, but now the map $\varphi^\#$ somehow gives us the wrong direction. It turns out we want to tensor $\mathcal{O}_X \otimes_{\varphi^{-1} \mathcal{O}_Y} \mathcal{O}_X =: \varphi^* \mathcal{N}$ and for modules, this is the adjoint of the direct image functor.

In this case, we have $\mathcal{M} = \tilde{M}$ for M an A-module. Write M^B for M considered as a B-module according to α. By definition,

$$\varphi_* (\mathcal{M}) (X_f) = \mathcal{M} \left(\varphi^{-1} (Y_f)\right) = \mathcal{M} \left(X_{\alpha(f)}\right) = \left(M_{\alpha(f)}\right)^{B_f} = \left(M^B\right)_f \ .$$

So we just have $(\varphi_* \mathcal{M}) (Y) = M^B = M$. The upshot of this is that

$$\varphi_* \left(\tilde{M}\right) = \widetilde{M^B} \ .$$

2. Arbitrary schemes

Unfortunately, for a morphism $\varphi : X \to Y$ of arbitrary schemes, it's not even true that it has to preserve qco sheaves.

Example 1. Recall that localization doesn't commute with infinite products. This means we can find a non-qco sheaf by taking an infinite product of qco sheaves. Let $Y = \operatorname{Spec} B$. We know $\tilde{B} = \mathcal{O}_Y$, and then we want to consider the (countably) infinite product $\mathcal{O}_Y^\infty$. Assume there is $f \in B$ which not a zero divisor, and not

Date: February 8, 2019.

a unit. We can also consider B^∞, and then form the sheaf $\widetilde{B^\infty}$. Taking global sections commutes with infinite products, so

$$\mathcal{O}_Y^\infty(Y) = B^\infty \ .$$

Now we want to look at how sections of this on Y_f compare. Consider

$$\left(1, f^{-1}, f^{-2}, \cdots\right) \in B_f^\infty = \mathcal{O}_Y^\infty(Y_f) \ .$$

Not however that this is not an element of $(B^\infty)_f$. If this were qco, it would have to be the sheaf associated to this module, but we just saw that it isn't. I.e. $\mathcal{O}_Y^\infty \neq \widetilde{B^\infty}$.

Now we want to get it as a direct image. But this is easy, just take

$$X = \coprod_\infty Y$$

to be an infinite disjoint union where φ is just projection. Then $\varphi^* \mathcal{O}_X$ is just $\mathcal{O}_Y^\infty$ so we found a qco sheaf which has direct image which is not qco.

This motivates the following definitions.

Definition 1. A space X is quasicompact if every open covering has a finite subcover.[1]

Example 2. Recall $X = \operatorname{Spec} R$ is quasicompact. Take some open cover

$$X = \bigcup U_\alpha$$

where $U_\alpha = X \setminus V(I_\alpha)$. But this is equivalent to

$$\emptyset = \bigcap V(I_\alpha) = V\left(\sum I_\alpha\right)$$

which means $\sum I_\alpha = (1)$, so we have some finite sum

$$1 = f_{\alpha_1} + \cdots + f_{\alpha_k}$$

and then we have a finite cover

$$X = \bigcup_{i=1}^k U_{\alpha_i} \ .$$

Example 3. The following is a slightly more interesting example. Consider $\mathbb{A}_k^\infty \setminus \{0\}$. This is covered by the

$$X_{x_i} = \operatorname{Spec} k\left[x_i^{-1}, x_1, \cdots\right] \ .$$

But the ideal of 0 is not finitely generated, so there can never be a finite subcover so this cannot be quasicompact.

Remark 1. Note that any space X is quasicompact iff X has a finite affine covering.

Definition 2. A morphism $\varphi : X \to Y$ is quasicompact if (equivalently)[2]

(i) For U quasicompact, $\varphi^{-1}(U)$ is quasicompact.
(ii) $\varphi^{-1}(U)$ is quasicompact for all affine open subsets $U \subset Y$.
(iii) $\varphi^{-1}(U)$ is quasicompact for U in some affine open covering of Y.

[1]Recall compact was exactly this, but also Hausdorff. When dealing with the Zariski topology Hausdorff spaces are very rare, so we don't want to insist on this.

[2]This is simultaneously a definition and a theorem saying these are equivalent.

Proof. $(i) \implies (ii) \implies (iii)$ is clear.

$(iii) \implies (i)$: Assume that $\varphi^{-1}(U)$ is quasicompact for U in some affine open covering of U. Then we want to show that the preimage of any quasicompact set is quasicompact. It is enough to show that the preimage of any $U_f = \operatorname{Spec} R_f$ is quasicompact. So let $U = \operatorname{Spec} R \subseteq Y$ be in some affine open covering of U such that $\varphi^{-1}(U)$ is quasicompact. This means

$$W = \varphi^{-1}(U) = W_{\alpha_1} \cup \cdots \cup W_{\alpha_k}$$

but then we have

$$\varphi^{-1}(U_f) = W_{\alpha_1,f} \cup \cdots \cup W_{\alpha_k,f}$$

so we are done. □

It would be nice if we could say that quasicompact morphisms preserve qco sheaves, but unfortunately this isn't true either, so we have to make another definition. For a morphism $\varphi : X \to Y$, we can consider $X \times_Y X$ and the universal property says that a morphism to this comes in the form of two Y-morphisms to X. In particular, we can consider the diagonal map Δ:

$$\begin{array}{c} X \xrightarrow{\Delta} X \times_Y X \\ (x) \longmapsto (x,x) \end{array}.$$

Definition 3. φ is *separated* if $\Delta(X)$ is closed in $X \times_Y X$.

Note that this implies that Δ is a closed embedding.

Definition 4. φ is quasi-separated if Δ is a quasi-compact morphism.

Let's get to know this definition a bit. Let $Y = \cup_\alpha Y_\alpha$ be covered by affine Y_α. Then $X = \cup_\alpha \varphi^{-1}(Y_\alpha)$ and then we have the diagonal map

$$\Delta : X_\alpha \to X_\alpha \times_Y X_\alpha = X_\alpha \times_{Y_\alpha} X_\alpha \ .$$

Now let $X \to Y = \operatorname{Spec} R$ and cover X with affines $X = \cup X_\beta$. Now the

$$X_\alpha \times_R X_\beta = \operatorname{Spec} S_\alpha \otimes_R S_\beta$$

cover $X \times_R X$ so $X_\alpha \hookrightarrow X_\alpha \times X_\alpha$ is really $\operatorname{Spec} S_\alpha \hookrightarrow \operatorname{Spec} S_\alpha \otimes_R S_\alpha$ which corresponds to $S_\alpha \otimes_R S_\alpha \twoheadrightarrow S_\alpha$, with kernel consisting of elements which look like $x \otimes 1 - 1 \otimes x$. This means the diagonal map $X_\alpha \hookrightarrow X_\alpha \times X_\alpha$ is a closed embedding. The issue is that we potentially don't know this is the case for X itself. But if the image is closed, and it is locally a closed embedding, then the whole thing is a closed embedding. The consequence of this is that if X is separated over an affine scheme like this, the intersection of two affines will be affines.

As it turns out, being quasicompact and quasiseparated together will be enough to preserve qco sheaves.