LECTURE 9
MATH 256B

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1. Separated and quasi-separated morphisms

Recall we had the following definitions. Let $f\colon X \to Y$ be a morphism of schemes. Then we can consider the diagonal map $\Delta\colon X \to X \times_Y X$. All properties of this map effectively reduce to the case of Y affine by some covering argument, so WLOG let Y be affine. Now we can cover X with affines

$$X = \bigcup_\alpha X_\alpha \tag{1}$$

and then the open piece

$$\bigcup_\alpha X_\alpha \times_Y X_\alpha \subseteq X \times_Y X \tag{2}$$

is enough to contain the image of Δ. Since the map $\Delta\colon X_\alpha \to X_\alpha \times_Y X_\alpha$ corresponds to the surjective ring homomorphism $R \otimes_S R \twoheadrightarrow R$ (where $X_\alpha = \operatorname{Spec} R$) we have that $\Delta X_\alpha \hookrightarrow X_\alpha \times_Y X_\alpha$ is a closed embedding. So the diagonal map is always locally closed.

Definition 1. f is *separated* if $\Delta(X)$ is closed.[1] We say f is *quasi-separated* if Δ is a quasicompact morphism. Note that separated implies quasi-separated.

Example 1. Let $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$. Then the diagonal map corresponds to the surjective ring homomorphism $B \otimes_A B \twoheadrightarrow B$. So the image is closed, and f is separated. More generally all affine morphisms are separated.

Definition 2. A scheme X is called *separated* (resp. *quasi-separated*) if the unique morphism $X \to \operatorname{Spec} \mathbb{Z}$ is separated (resp. quasi-separated).

Consider any $X \to \operatorname{Spec} R = Y$. Now we can always compare $X \times_Y X$ to $X \times_{\operatorname{Spec}\mathbb{Z}} X$. We always have a map $X \times_Y X \to X \times_{\operatorname{Spec}\mathbb{Z}} X$. Notice that, in general, $A \otimes_R B$ is a quotient of $A \otimes_{\mathbb{Z}} B$ by the additional relations of being linear with respect to R rather than just $\mathbb{Z}$. This means that the map $X \times_Y X \to X \times_{\operatorname{Spec}\mathbb{Z}} X$ is a closed embedding:

$$\begin{array}{ccc} & & X \times_Y X \\ & \overset{\Delta_Y}{\nearrow} & \big\downarrow \\ X & & \\ & \underset{\Delta Y}{\searrow} & \\ & & X \times_{\operatorname{Spec}\mathbb{Z}} X \end{array} \quad . \tag{3}$$

Date: February 11, 2019.

[1] Really we define this to be that Δ must be a closed embedding, but by the preceding argument, being closed actually implies it is a closed embedding, since it is always locally a closed embedding.

This means that each diagonal map is closed exactly when the other one is, and similarly they are quasi-separated when the other is. Therefore the map $X \to \operatorname{Spec} \mathbb{Z}$ being separated/quasi-separated is equivalent to $X \to Y = \operatorname{Spec} R$ being separated/quasi-separated.

Proposition 1. *If X is separated, $U, V \subset X$ being open and affine implies $U \cap V$ is affine.*

Remark 1. The converse to this is false.

Proof. The reason for this is almost trivial:

$$\begin{array}{ccc} X & \xrightarrow{\Delta} & X \times_Y X \\ & & \uparrow \\ U \cap V & \xrightarrow{\Delta} & U \times_Y V \end{array} \quad . \tag{4}$$

□

Proposition 2. *If X is quasi-separated and $U, V \subset X$ are open affines, then $U \cap V$ is quasicompact.*

The proof for this is the same. Note however that this is iff.

Example 2. Consider the affine line $\mathbb{A}^1_k$ and an open subset $U = \mathbb{A}^1_k \setminus \{0\}$. Now we want to glue two copies of $\mathbb{A}^1_k$ along U to get

(5) ,

which we call X. Then we have the map $f : X \to Y = \mathbb{A}^1_k$ and we can consider the diagonal map $X \to X \times_Y X$ which looks like

(6)

but this is not a closed embedding, so f is not a separated morphism. It is however quasi-separated. Note however that the intersection of two affines is affine, which means this is a counterexample to the converse of Proposition 1.

Example 3. Consider the affine plane $\mathbb{A}^2_k$ and glue two copies of the plane with the origin doubled as before. But the intersection of the two affines that we glued together is not affine, so this tells us that it is not separated by Proposition 1.

Example 4. If we take $\mathbb{A}^\infty_k = \operatorname{Spec}[x_1, \cdots]$, remove the origin to get $U = \mathbb{A}^\infty_k \setminus V(x_1, x_2, \cdots)$. Note this is not quasi-compact. Now glue two copies together along this to get X, i.e. $\mathbb{A}^\infty_k$ with the origin doubled. So it is covered by two affines, but the intersection of these affines is not even quasi-compact so the morphism $X \to \mathbb{A}^\infty_k$ (or equivalently just $X \to \operatorname{Spec} \mathbb{Z}$) is not quasi-separated.

2. Quasi-coherent sheaves

Now we see how these definitions apply to sheaf theory.

Theorem 3. *If $\varphi \colon X \to Y$ is quasicompact and quasi-separated, then $\varphi_* (\mathbf{QCoh}(X)) \subseteq \mathbf{QCoh}(Y)$.*

Proof. Let $\mathcal{M} \in \mathbf{QCoh}(X)$. WLOG let Y be affine since we could just take the preimage of an affine cover and it would reduce to this. Now we take $\varphi_*(\mathcal{M}) = \tilde{N}$

and we want it to be the sheaf associated to some module, but we know this has to be $N = \mathcal{M}(X)$. So this question is really asking if

$$\mathcal{M}(X_f) = \mathcal{M}(X)_f \ . \tag{7}$$

Consider an affine open covering $X = \bigcup X_i$. WLOG we can take this to be finite since X is quasi-compact. Now if X is quasi-separated, we can assume that

$$X_i \cap X_j = \bigcup_k X_{ijk} \tag{8}$$

where the X_{ijk} are affine, and there are only finitely many of them.

Now we want to understand $\mathcal{M}(X)$. By the sheaf axiom we get a map to its restrictions, and in fact we get an exact sequence:

$$0 \to \mathcal{M}(X) \to \bigoplus_i \mathcal{M}(X_i) \to \bigoplus_{ijk} \mathcal{M}(X_{ijk}) \ . \tag{9}$$

Then we have the same for X_f:

$$0 \to \mathcal{M}(X_f) \to \bigoplus_i \mathcal{M}(X_{i,f}) \to \bigoplus_{i,j,k} \mathcal{M}(X_{ijk,f}) \ . \tag{10}$$

The final two terms are just localization at f:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{M}(X) & \longrightarrow & \bigoplus_i \mathcal{M}(X_i) & \longrightarrow & \bigoplus_{ijk} \mathcal{M}(X_{ijk}) \\ & & & & \downarrow (-)_f & & \downarrow (-)_f \\ 0 & \longrightarrow & \mathcal{M}(X_f) & \longrightarrow & \bigoplus_i \mathcal{M}(X_{i,f}) & \longrightarrow & \bigoplus_{i,j,k} \mathcal{M}(X_{ijk,f}) \end{array} \ . \tag{11}$$

Since localization is an exact functor, we have $\mathcal{M}(X) \to \mathcal{M}(X_f) = \mathcal{M}(X)_f$. □

This looks a lot like the proof that $\mathcal{M} \in \mathbf{QCoh}(X)$ for affine X implies that $\mathcal{M} = \tilde{M}$. In fact this can be slightly generalized:

Lemma 4. *Let $\mathcal{M} \in \mathbf{QCoh}(X)$. If X is quasi-compact and quasi-separated, and $\mathcal{M} \in \mathbf{QCoh}(X)$, then $f \in \mathcal{O}_X(X)$ implies $\mathcal{M}(X_f) \cong \mathcal{M}(X)_f$.*

So if we take X to be affine we get the theorem from a few lectures ago, and if we take it to be the preimage under such a morphism as in Theorem 3, we get Theorem 3.

3. Inverse images and tensor products

For $\varphi\colon X \to Y$ and $\mathcal{M}$ an $\mathcal{O}_X$-module, then $\varphi^{-1}\mathcal{M}$ is a $\varphi^{-1}\mathcal{O}_Y$-module, but $\varphi^{\#}\colon \varphi^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ sort of goes in the wrong direction. So we define

$$\varphi^*\mathcal{M} \coloneqq \mathcal{O}_X \otimes_{\varphi^{-1}\mathcal{O}_y} \varphi^{-1}\mathcal{M} \tag{12}$$

which turns out to be the fix by its universal property, but we should think about what this really means.

Let X be any space, $\mathcal{A}$ be a sheaf of rings, and $\mathcal{M}$ and $\mathcal{N}$ sheaves of $\mathcal{A}$-modules. Then we want to form $\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}$. First consider the presheaf tensor product

$$(\mathcal{M} \otimes_{\mathcal{A}}^{pr} \mathcal{N})(U) = \mathcal{M}(U) \otimes_{\mathcal{A}(U)} \mathcal{N}(U) \tag{13}$$

but the issue is that this isn't a sheaf. So we sheafify it to get

$$\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N} \coloneqq \mathrm{sh}(\mathcal{M} \otimes_{\mathcal{A}}^{pr} \mathcal{N}) \ . \tag{14}$$

The conclusion will basically be that this has the correct stalks:

$$(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N})_p = \mathcal{M}_p \otimes_{\mathcal{A}_p} \mathcal{N}_p \tag{15}$$

and the correct universal property:

$$\begin{array}{ccc} \mathcal{M} \times \mathcal{N} & \xrightarrow{\text{bilinear}} & \mathcal{L} \\ {\scriptstyle -\otimes-}\downarrow & \overset{\exists!}{\nearrow} & \\ \mathcal{M} \otimes \mathcal{N} & & \end{array} \quad . \tag{16}$$