LECTURE 1
MATH 261

LECTURES BY: PROFESSOR DAVID NADLER
NOTES BY: JACKSON VAN DYKE

The text is Varadarajan's Lie groups, Lie algebras and their representations. We won't follow this closely. Office hours haven't been specified yet.

1. Manifolds

1.1. **Preliminaries.** We will start with a review of smooth manifolds.

Definition 1. A manifold is a pair $\left(M, \{\mathcal{U}_\alpha, \varphi_\alpha\}_{\alpha\in A}\right)$ where M is a topological space, and $\{\mathcal{U}_\alpha, \varphi_\alpha\}$ is an atlas. This means the $\mathcal{U}_\alpha$ are open subsets and

$$\varphi_\alpha : \mathcal{U}_\alpha \xrightarrow{\sim} V_\alpha \subset \mathbb{R}^n$$

are homeomorphisms. These must satisfy the properties:

(1) The $\mathcal{U}_\alpha$ cover M.
(2) Compatibility in the sense that $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha (U_\alpha \cap U_\beta) \to \varphi_\beta (\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$ is smooth.
(3) Maximality.

Example 1. Consider the n-sphere

$$S^n = \left\{x \in \mathbb{R}^n \mid |x|^2 = 1\right\}$$

We have effectively glued two copies of $\mathbb{R}^n$ according to open embeddings.

If we go about making spaces by gluing things together, we get two sort of pathologies. The first is that the resulting space will not always be Hausdorff.

Example 2. Consider $M = \mathbb{R} \amalg_{\mathbb{R}^\times} \mathbb{R}$. That is we take the disjoint union of two copies of $\mathbb{R}$ glued along the nonzero values in $\mathbb{R}$. This can be imagined as $\mathbb{R}$ with two zeros. The basic problem here is that continuous functions can't tell the difference between these two points.

Remark 1. Note that we can perform this gluing in two different ways. If we embed $\mathbb{R}^\times$ with the identity, we get this pathological non-Hausdorff space. If we however take one embedding to be the identity, and the other embedding to be the inverse function, then you get a different space. In fact this second space turns out to be S^1.

This motivates some typical additional assumptions:

(1) M is Hausdorff.
(2) M is paracompact.

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Paracompactness just means that every time we have an open cover, we can find a refinement of this cover such that the resulting open cover is locally finite. A typical example of what this condition restricts us from considering, i.e. something which is not paracompact, is the long line. The main reason that we like these conditions, is that we want well-behaved function theory. For example we want partitions of unity subordinate to covers.

For this class, all n-manifolds will be closed submanifolds of $\mathbb{R}^N$ for some $n \ll N$.

1.2. Categorical point of view.

There is a category **Mfd** where the objects are manifolds, and between any two manifolds, we have the set $\operatorname{Hom}_{\mathbf{Mfd}}(M, N)$ which consists of the smooth maps $M \to N$. Note that we have compositions

$$\operatorname{Hom}(M, N) \times \operatorname{Hom}(N, P) \to \operatorname{Hom}(M, P)$$

given by the set-theoretic composition.

Note that there are many flavors:

(1) Smooth manifolds (smooth)
(2) Complex manifolds (holomorphic)
(3) Smooth algebraic varieties (polynomial maps)
(4) Banach manifolds (smooth)

The point is, there are lots of contexts where it makes sense to talk about a manifold, and these contexts are characterized by a particular notion of a "good" function.

2. Lie groups

Definition 2. A *Lie group* G is a group object in **Mfd**.

This means the following:

(1) G is a manifold
(2) G is a group
(3) These structures are compatible in the sense that multiplication $G \times G \xrightarrow{m} G$ and inverse $G \xrightarrow{i} G$ are smooth.

Recall that this means

(1) m is associative
(2) There is a unit $e \in G$ such that

$$m(g, e) = m(e, g) = g \qquad m(g, i(g)) = e = m(i(g), g)$$

Exercise 1. Derive that i is smooth from the fact that m is smooth.

Example 3. S^1 is a Lie group in the sense that

$$S^1 = \mathbb{R}/\mathbb{Z}$$

where the multiplication is just addition. Via an exponential map we can also write this as

$$\mathbb{R}/\mathbb{Z} = \{z \in \mathbb{Z} \mid |z| = 1\}$$

We can also regard $S^1 \hookrightarrow \mathbb{C}^\times$ as a subset of the nonzero complex numbers, which is also a Lie group. We say S^1 is a *Lie subgroup.*

Example 4. Now consider $S^3 \subset \mathbb{R}^4$ where we think of $\mathbb{R}^4 = \mathbb{H}$ as the quaternions. Recall that these are

$$\mathbb{H} = \{a + ib + jc + kd \mid a, b, c, d \in \mathbb{R}\}$$

with the rules that:

$$i^2 = j^2 = k^2 = -1 \qquad ij = k \qquad jk = i \qquad ki = j$$

Note that the unit is just 1 and the inverse of $a + bi + cj + dk$ is

$$\frac{a - bi - cj - dk}{a^2 + b^2 + c^2 + d^2}$$

Now if we look at $S^3 \hookrightarrow \mathbb{H}$, it turns out that we have the following:

Exercise 2. Check that S^3 is closed under quaternionic multiplication.

So S^3 is a Lie group because of the quaternions.

Example 5. As a Lie group, $\mathbb{C}^\times \simeq S^1 \times \mathbb{R}_{>0}$ where we send $re^{i\theta}$ to (θ, r) and similarly, $\mathbb{H}^\times \simeq S^3 \times \mathbb{R}_{>0}$. Note that $\mathbb{C}^\times$ and $\mathbb{H}^\times$ are noncompact. Note that this has nothing to do with the group theory, but rather the geometry. Similarly, we can talk about abelian Lie groups, but this is independent of the manifold structure.

Remark 2. None of the other spheres besides S^0, S^1, and S^3 are Lie groups. We might expect S^7 to be a Lie group on account of the so-called octonions, but the octonions do not form an associative algebra.

Example 6. S^2 is not a Lie group. Recall that $\chi\left(S^2\right) = 2$. But we also have the following:

Claim 1. If G is a connected Lie group, $\chi(G) = 0$. If G is finite, $\chi(G) = |G|$.

Example 7. Vector spaces and their automorphisms provide some nice[1] examples. Let V be a finite dimensional vector space, then the general linear group:

$$\mathrm{GL}(V) = \mathrm{GL}(n, \mathbb{R}) = \mathrm{Aut}(V)$$

is the set of $n \times n$ invertible matrices.

Now recall that $\mathrm{GL}(V)$ also acts on $\Lambda^{\dim V} V \simeq \mathbb{R}$, and we get the following

$$\begin{tikzcd}
\mathrm{GL}(V) \arrow[d, "\det"] & \circlearrowright & V \\
\mathrm{GL}\left(\Lambda^{\dim V} V\right) & \circlearrowright & \Lambda^{\dim V} V \simeq \mathbb{R}
\end{tikzcd}$$

so we get a short exact sequence:

$$1 \to \mathrm{SL}(V) \to \mathrm{GL}(V) \to \mathrm{GL}\left(\Lambda^{\dim V} V\right) \to 1$$

So the reason some people "don't like" $\mathrm{GL}(V)$ is that it has a normal subgroup, so it is not simple. Note that it splits as a manifold, but not as a group. But is $\mathrm{SL}(V)$ simple? If a group has a nontrivial center, then it certainly isn't simple. And the center of $\mathrm{SL}(V)$ has the diagonal matrices

$$\begin{pmatrix} a & \cdots & 0 \\ \cdots & a & \cdots \\ 0 & \cdots & a \end{pmatrix}$$

where $a^n = 1$ as its center.

Exercise 3. Show that $[\mathrm{SL}(V), \mathrm{SL}(V)] = \mathrm{SL}(V)$.

Fact 1. *All normal subgroups of* $\mathrm{SL}(V)$ *are finite, and in fact are contained in the center.*

[1] According to Professor Nadler, humans understand linear things and basically nothing else.

2.1. **Group actions.** Group actions are the main reason Lie was interested in such things to begin with.

Definition 3. Given a Lie group G and a manifold X, an action of G on X is a smooth map $a : G \times X \to X$ such that

(1) a is associative:
$$a\left(g_1, a\left(g_2, x\right)\right) = a\left(m\left(g_1, g_2\right), x\right)$$
(2) This is unital, so $a\left(e, x\right) = x$.

The two examples to think about are the following:

Example 8. Suppose X is a vector space, and every element of G acts by not only a smooth map, but a linear map. Then this action is called a *representation* and we think of this as a homomorphism $G \to \mathrm{GL}\left(X\right)$.

Exercise 4. Understand the natural action of $\mathrm{SL}\left(2, \mathbb{C}\right)$ on $\mathbb{CP}^1$.

LECTURE 2
MATH 261A

LECTURES BY: PROFESSOR DAVID NADLER
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1. Examples of Lie groups and representations

Recall that a Lie group is a smooth manifold G, which is also a group such that the group multiplication and inverse map is smooth with respect to the manifold structure. These of course have to be associative and unital.

Also recall the nature of a group action on a space. We will always have in mind that the space we are acting on is some smooth manifold X. The action is a smooth map $G \times X \to X$. This action must also satisfy associativity and that the identity acts as the identity diffeomorphism.

We should keep the following examples in mind.

Example 1. The group $G = \mathrm{GL}(n, \mathbb{C})$ is a Lie group consisting of $n \times n$ invertible matrices.

Example 2. A representation is a special case of a group action on a manifold. For any vector space V, $G \times V \to V$ is given by linear diffeomorphisms which are of course associative and unital.

Example 3. In particular, consider $\mathrm{GL}(2, \mathbb{C})$. This consists of all matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

such that $ad - bc \neq 0$. Take $X = \mathbb{CP}^1$. The action is $\mathrm{GL}(2, \mathbb{C}) \times \mathbb{CP}^1 \to \mathbb{CP}^1$ which maps

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, l = \begin{pmatrix} x \\ y \end{pmatrix} \qquad \mapsto \qquad \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

If we think about this in terms of slope, this says that the line with slope y/x goes to the line with slope $(cx + dy) / (ax + by)$. This is what is called a fractional linear transformation.

Example 4. For $V = \mathbb{C}^2$ we get an example of a representation where $\mathrm{GL}(2, \mathbb{C}) \times \mathbb{C}^2 \to \mathbb{C}^2$ is the natural action.

2. Higher dimensional examples

We now generalize to any n. For $G = \mathrm{GL}(n, \mathbb{C})$, $V = \mathbb{C}^n$, what are the potential spaces X we might consider? We will consider complex projective space, the Grassmannian of n planes, and flag manifolds.

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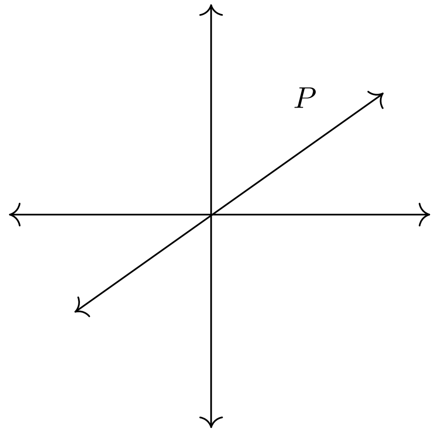

FIGURE 1. The horizontal axis is $\mathbb{C}^k$, and the vertical axis is $\mathbb{C}^{n-k}$. The line P is such that $P \cap \mathbb{C}^{n-k} = \{0\}$.

2.1. **Complex projective space.** Well first we can have $X = \mathbb{CP}^{n-1}$.

Exercise 1. What is the analogue of the "slope" in this higher dimensional case?

How do we see this is a manifold? We cover this with copies of $\mathbb{C}^{n-1}$. For all $i \in \{1, \cdots, n\}$ write

$$\mathcal{U}_i \simeq \mathbb{C}^{n-1} \simeq \{(x_1, \cdots, x_{i-1}, 1, x_{i+1}, \cdots, x_n)\}$$

Exercise 2. Check that these are all appropriately compatible in their intersections.

2.2. **Grassmannian.** Another possibility, is to consider the "Grassmannian" which is

$$X = \mathrm{Gr}\,(k, n, \mathbb{C}) = \{k\text{-planes in } \mathbb{C}^n \text{ through } 0\}$$

Exercise 3. For what k,k',n,n' do we have a diffeomorphism between $\mathrm{Gr}\,(k, n, \mathbb{C}) \simeq \mathrm{Gr}\,(k', n', \mathbb{C})$.

How do we see this is a manifold? Consider the following chart in $\mathrm{Gr}\,(k, n)$. Consider a k-plane P as in fig. 1. such that $P \cap \mathbb{C}^{n-k} = \{0\}$. Then consider $\mathcal{U}$ to be the set of all such k-planes. Note that $\mathcal{U} \simeq \mathrm{Hom}\left(\mathbb{C}^k, \mathbb{C}^{n-k}\right)$. This is of course just a collection of matrices, so $\mathcal{U} \simeq \mathbb{C}^{k(n-k)}$. Now we need to check that these objects actually cover $\mathrm{Gr}\,(n, k)$. We will take two approaches.

This open set $\mathcal{U}$ can be defined anytime we break this up into k coordinates, and the complement. That is, for any $I \subset \{1, \cdots, n\}$ such that $|I| = k$, we can split this space into $\mathbb{C}^I$ and $\mathbb{C}^{I^c}$ and now we can define $\mathcal{U}_I$ for this choice of I.

The second approach is to note the following:

Exercise 4. $\mathrm{GL}\,(n, \mathbb{C})$ acts transitively on all three of the spaces we are considering here.

Then we can use the group action to move $\mathcal{U}$ around to cover.

Exercise 5. Check that these agree on the intersections.

2.3. **Flag manifolds.** Another example is a flag manifold. Let's write some subset

$$\underline{k} \subseteq \{1, \cdots, n-1\} \ .$$

Then we can consider the flag manifold $\mathrm{Fl}(k)$ which consists of nested sequences of subspaces $E_{k_1} \subset E_{k_2} \subset \cdots$ with dimension of $E_{k_i} = k_i$ where k_i is the ith element of $\underline{k}$.

To see this is a manifold, we can consider it as a subspace of the following:

$$\mathrm{Fl}(\underline{k}, n, \mathbb{C}) \subseteq \prod_{k_i} \mathrm{Gr}(k_i, n, \mathbb{C})$$

Exercise 6. Show that $\mathrm{Fl}(\underline{k}, n, \mathbb{C})$ is cut out of the above as a regular value of a smooth map so it is a submanifold.

3. Types of group actions

We now introduce some terminology for different types of group action. We will write an action $G \times X \to X$ as $G \circlearrowright X$. We say an action is transitive if for every $x, y \in X$, there exists some $g \in G$ such that $g \cdot x = y$. This is somehow saying G is bigger than X. We can also ask that the action is free, which means for every $x \in X$, if $g \cdot x = x$, then $g = e$. This is somehow saying X is bigger than G.

Define the orbit of $x \in X$ to be

$$X \supseteq G \cdot x \coloneqq \{y \in X \mid \exists g \in G \text{ s.t. } y = g \cdot x\}$$

Then the stabilizer is

$$G \supseteq G_x \coloneqq \{g \in G \mid g \cdot x = x\}$$

Note that an action is transitive iff there is only one orbit, and an action is free iff every stabilizer is trivial.

Lemma 1. *Stabilizers are closed subgroups. In addition, for $y = gx$, we have $G_y = gG_xg^{-1}$.*

Proof. The second statement is effectively obvious so we focus on the first statement. The fact that the stabilizer is a subgroup is immediate. We prove it is closed. The stabilizer G_x is the fiber at x of the map $g \mapsto g \cdot x$. Since X is Hausdorff, points are closed, so the fiber is closed, so the stabilizer is closed. □

Example 5. Consider $G = \mathrm{GL}(n, \mathbb{C}) \circlearrowright \mathrm{Fl}(\underline{k}, n, \mathbb{C})$. This is a transitive action so there is only one orbit. The stabilizer G_x of a point

$$x = \{E_{k_1} = \mathrm{Span}\{e_1, \cdots, e_{k_1}\}, \cdots, E_{k_i} = \mathrm{Span}\{e_1, \cdots e_{k_i}\}\}$$

is the collection of matrices such that the top left k_i block has zeros beneath it for every i. Note that for full flags we get the collection of all upper-triangular matrices in $\mathrm{GL}(n, \mathbb{C})$.

Based on lemma 1 we introduce the following definition:

Definition 1. A *Lie subgroup* $H \subset G$ is a subgroup, which is also closed.

Example 6 (Non-example)**.** We offer a subgroup $H \subset G$ which is not closed. Take $G = T^2$, and then $H = \mathbb{R} \times \{\text{irrational slope}\}$ then we get a subgroup which is not closed.

From now on we assume all subgroups are Lie subgroups.

Lemma 2. *Lie subgroups are Lie groups. In particular we have a bijection between Lie subgroups and transitive G actions.*

Exercise 7. Prove lemma 2. I.e. show that Lie subgroups are in fact submanifolds.

Example 7. Consider $\mathrm{GL}(2,\mathbb{C}) \circlearrowleft \left(\mathbb{CP}^1\right)^k$. For what k is this transitive? For what k is this free? For what k does this have finitely many orbits? What are the stabilizers?

The case $k = 0$ is trivial. For $k = 1$ this is transitive but not free. It is not free because the diagonal matrices scale the vectors without changing the line, so the stabilizer of any point contains the diagonal matrices which comprise $\mathbb{C}^\times$. For $x = (1, 0)$,

$$G_x = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mid ad \neq 0 \right\}$$

For $k = 2$, this action is not transitive, and it has two orbits which consist of pairs of matching lines $l_1 = l_2$, and different lines $l_1 \neq l_2$. What about the stabilizers? First take x to be $l_1 = l_2 = [e_1]$, and we get

$$G_x = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mid ad \neq 0 \right\}$$

and then for x consisting of $l_1 = [e_1]$,$l_2 = [e_2]$ we get

$$G_x = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \mid ad \neq 0 \right\}$$

Exercise 8. Complete the same analysis for $k = 3$.

Exercise 9. Consider $\mathrm{GL}(2,\mathbb{R}) \circlearrowleft \mathbb{CP}^1$. Calculate the orbits. Calculate the stabilizers.

Next time we start with G acting on itself by left/right translations. This will lead us to Lie algebras.

LECTURE 3
MATH 261A

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Office hours are now settled to be after class on Thursdays from $12:30-2$ in Evans 815, or still by appointment.

1. The action of $\mathrm{GL}(2,\mathbb{C})$ on $\left(\mathbb{CP}^1\right)^k$

Recall we are studying the action of $\mathrm{GL}(2,\mathbb{C})$ on $\left(\mathbb{CP}^1\right)^k$. We already thought about $k=0,1,2$. When $k=3$, we are studying triples of lines in $\mathbb{C}^2$. There are three orbits of this action. The first is when $l_1=l_2=l_3$. This looks like a copy of $\mathbb{CP}^1$ again. The stabilizer of the configuration $l_1=l_2=l_3=e_1$, consists of upper triangular matrices

$$B=\left\langle\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\right\rangle$$

Note that $\mathbb{CP}^1\simeq \mathrm{GL}(2,\mathbb{C})/B$. B is for Borel subgroup. Note B is not Abelian.

Then $l_i=l_j\neq l_k$ is another orbit, which looks like $\left(\mathbb{CP}^1\times\mathbb{CP}^1\right)\setminus\mathbb{CP}^1$ diagonal , so we just remove the diagonal. For $l_i=l_j=e_1$ and $l_j=e_2$, the stabilizer consists of diagonal matrices

$$T=\left\langle\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}\right\rangle$$

Note that, the orbit $\left(\mathbb{CP}^1\times\mathbb{CP}^1\right)\setminus\mathbb{CP}^1\simeq \mathrm{GL}(2,\mathbb{C})/T$. T is for torus.

The final orbit consists of distinct lines. This is an open, dense, orbit. This is all that's left, so it's $\left(\mathbb{CP}^1\right)^3$ minus everything else. The stabilizer of $l_1=e_1$, $l_2=e_2$ and $l_3=(1,1)$ is Z for center consisting of

$$Z=\left\langle\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right\rangle$$

For $a\neq 0$.

Note that we have the following exact sequence:

$$\begin{tikzcd} 1 \arrow[r] & \mathbb{C} \arrow[r] & B \arrow[r] & T \arrow[r] \arrow[l, bend right] & 1 \end{tikzcd}$$

So we can write this as a semidirect product $B\simeq\mathbb{C}\rtimes T$.

Exercise 1. Describe $T\circlearrowright\mathbb{C}$.

The third orbit is just $\mathcal{O}=\mathrm{GL}(2,\mathbb{C})/Z$. Since Z is the center, it is normal, which means this is a group. This has a name: $\mathrm{PGL}(2,\mathbb{C})\simeq\mathrm{GL}(2,\mathbb{C})/Z$.

Exercise 2. We know $\mathrm{GL}(n,\mathbb{C})=\mathrm{Aut}_{\mathrm{Vect}}(\mathbb{C}^n)$. Convince yourself that $\mathrm{PGL}(n,\mathbb{C})=\mathrm{Aut}_{\mathbf{AlgVar}}\left(\mathbb{P}^{n-1}\right)$.

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Assume a Lie group $G \circlearrowright X$ simply transitively, i.e. transitive and free. Then we might hope that X is also a group, or that they're canonically isomorphic. But the point is, they are not until you choose a point in $x \in X$. This point is then the identity. The way this is an isomorphism, is $G \to X$ where $g \mapsto g \cdot x$. In this situation, we call this a principal G bundle over a point, or a G-torsor.

Example 1. One example is V a vector space and X an affine space modeled on V. This doesn't have an origin a priori.

2. More manifold review

Now we start differentiating. Let's review tangent and cotangent bundles. Recall the category **Mfd** where manifolds are objects, and the morphisms are smooth maps between them. Then we have a tangent bundle functor $T : \mathbf{Mfd} \to \mathbf{Mfd}$. Dually, there is a cotangent bundle which maps $T^* : \mathbf{Mfd} \to \mathbf{SymplMfd}$.

2.1. Tangent bundles.

Recall for M a smooth n-manifold, we have a rank n vector bundle $\pi : TM \to M$. If we regard $M \subset \mathbb{R}^N$ as living in an ambient $\mathbb{R}^N$, there is a copy of $\mathbb{R}^N$ living at every point $x \in M$, and then we can consider the subspace of this space which consists of vectors tangent to this point on the manifold. Furthermore, if $M = \{F = 0\}$, for $F : \mathbb{R}^N \to \mathbb{R}^{N-n}$, we have $M = F^{-1}(0)$ for 0 a regular value. In this space, the tangent space is the kernel of dF_x.

Observe that $TM \subset \mathbb{R}^N \times \mathbb{R}^N$ where the first copy consists of the points x, and the second copy consists of tangent vectors. So this whole thing is cut out by $F = 0$ and $dF = 0$.

2.2. Cotangent bundles.

A cotangent bundle $\pi : T^*M \to M$ consists of the dual space of these tangent space at each point. Note that T^* is a functor into symplectic manifolds rather than ordinary manifolds.

Consider a smooth map $f : M \to N$. If we want we can pass to the tangent bundles and get df. What we get here is a correspondence:

$$T^*M \xleftarrow[df^*]{} T^*N|_M = T^*N \times_N M \longrightarrow T^*N$$

Exercise 3. T^* is a functor.

Recall the composition of correspondences goes like this: If we have spaces X and Y and some correspondence C between them, along with a correspondence D between Y and Z. Then we form the fiber product $C \times_Y D$.

$$\begin{tikzcd} & & C \times_Y D \arrow[dl] \arrow[dr] & & \\ & C \arrow[dl] \arrow[dr] & & D \arrow[dl] \arrow[dr] & \\ X & & Y & & Z \end{tikzcd}$$

In fact not only are the manifolds that we end up with symplectic, but the correspondences are Lagrangian correspondences. Recall a symplectic manifold consists of a pair (X, ω) where ω is a 2-form which is closed and nondegenerate.

Alternatively we can view this as a section $\omega \in \Gamma\left(X, \Lambda^2 T^* X\right)$. All symplectic forms locally look like

$$\omega_0 = \sum_{i=1}^{n} dp_i \wedge dq_i$$

Any time we consider a correspondence, we can think of $T^*N|_M$, which can be thought of as a subset of the actual product (it is a fiber product after all). In fact it is even a submanifold:

$$L = T^*N|_M \hookrightarrow T^*M \times T^*N$$

Even better than this, it is Lagrangian, so $\omega|_L = 0$. Note that the symplectic form on $T^*M \times T^*N$ is $-\omega_M + \omega_N$.

3. Lie algebras

Consider the following question: what algebraic structure do vector fields on a manifold have? The whole point is that everything we will end up doing will be analogous to this. Write

$$\mathrm{Vect}\,(M) = \Gamma\,(M, TM)$$

The first thing we learn in a manifolds course is that this has a Lie bracket defined as follows: for two vector fields v, w, we can apply the following to a function:

$$[v, w]\, f = (vw - wv)\, f$$

Exercise 4. Show $[v, w]$ is also a vector field.

Solution. We can write:

$$v = \sum g_i \partial_{x_i} \qquad\qquad w = \sum_{i=1}^{n} h_i \partial_{x_i}$$

in some local coordinates. Then

$$[v, w]\, f = \left(\sum_i g_i \partial_{x_i}\right)\left(\sum_j h_j \partial_{x_j}\right) f - \left(\sum_j h_j \partial_{x_j}\right)\left(\sum_i g_i \partial_{x_i}\right) f$$

Then we have the identity that

$$\partial_{x_i} p = p \partial_{x_i} + p_i$$

for any function p. Now we want to move every derivative to the right, and the only remaining terms are first order.

Recall a Lie bracket satisfies several properties:

(1) Bilinearity over $\mathbb{R}$
(2) Skew-symmetric: $[v, w] = -[w, v]$
(3) Jacobi identity: $[u, [v, w]] = [[u, v], w] + [v, [u, w]]$

We can think of the Jacobi identity as a sort of Leibniz rule.

Fix a field k.

Definition 1. A Lie algebra over k is a k-vector space equipped with a bracket $[\cdot, \cdot] : V \otimes V \to V$ satisfying the three properties from above.

A morphism of Lie algebras

$$\varphi : (V_1, [\cdot,\cdot]_1) \to (V_2, [\cdot,\cdot]_2)$$

is a linear map such that $[\varphi v, \varphi w] = \varphi([v,w])$. In a certain imprecise sense, these can all be thought of as vector fields. There are two "standard" sources of Lie algebras.

Any time we have A such that A/k is an associative algebra, we can consider derivations of A, written $\mathrm{Der}(A)$. This consists of maps $d : A \to A$ which are k-linear and satisfy the Leibniz rule:

$$\partial(ab) = \partial a \cdot b + a \cdot \partial b$$

Exercise 5. Show that the composition of two derivations is not a derivation, but the bracket of two derivations is a derivation.

Exercise 6.
a. Show that $\mathrm{Der}(A)$ is a Lie algebra with a natural Lie bracket.
b. Check that $A = \mathcal{C}^\infty(M)$ is a commutative algebra, and

$$\mathrm{Vect}(M) = \mathrm{Der}(\mathcal{C}^\infty(M))$$

The other source is the following. Whenever D/k is an associative algebra, we get a Lie algebra D, where the bracket of two elements is just

$$[d_1, d_2] = d_1 d_2 - d_2 d_1$$

Exercise 7.
a. Show that D is a Lie algebra.
b. Consider $D = \mathrm{Diff}(M)$, the associative algebra of differential operators. This can be viewed as living inside $\mathrm{End}(\mathcal{C}^\infty(M))$. There is a natural filtration, where $\mathrm{Diff}^0(M) = \mathcal{C}^\infty(M)$. I.e. multiplying by any function is a differential operator. The elements p are characterized by satisfying $[p,f] = 0$ for any $f \in \mathcal{C}^\infty(M)$. Then $\mathrm{Diff}^{\leq 1}(M) \supseteq \mathrm{Diff}^0(M)$ which are characterized by $[p,f] \in \mathrm{Diff}^0(M)$ and so-on.

Then we have a map of Lie algebras

$$\mathrm{Vect}(M) \to \mathrm{Diff}(M)$$

and the image lands in $\mathrm{Diff}^{\leq 1}(M)$. In fact $\mathrm{Diff}^{\leq 1}(M)$ forms a Lie algebra with quotient and sub-algebra:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{C}^\infty(M) \arrow[r] & \mathrm{Diff}^{\leq 1}(M) \arrow[r] & \mathrm{Vect}(M) \arrow[r] \arrow[l, bend right] & 0 \end{tikzcd}$$

Check that this is all the case.

4. Relationship between associative algebras and Lie algebras

There is a pair of adjoint functors:

$$\begin{tikzcd} \mathbf{Lie}_k \arrow[r, bend left, "\mathcal{U}"] & \mathbf{Alg}_k \arrow[l, bend left, "\text{Forget}"] \end{tikzcd}$$

where the left adjoint to $F = \text{Forget}$ brings an algebra to the universal enveloping algebra. This means

$$\mathrm{Hom}_{\mathbf{Alg}}(\mathcal{U}V, A) = \mathrm{Hom}_{\mathbf{Lie}}(V, F(A))$$

Warning 1. One might hope that now it is the case that differential operators are the enveloping algebra of vector fields but this is not true in general.

Exercise 8. What is the relationship between $\operatorname{Diff}(M)$ and $\mathcal{U}(\operatorname{Vect} M)$.

5. Preview of relationship between Lie algebras and groups

Next time we will return to Lie groups, and in particular their relationship to Lie algebras. Lie algebras should roughly be viewed as derivations, e.g. vector fields, and similarly Lie groups should roughly be seen as morphisms, e.g. diffeomorphisms. Then groups give us big symmetries, and vector fields give us infinitesimal symmetries. In particular, the Lie algebra $\mathfrak{g}$ associated to a Lie group G is somehow a tangent plane to G at the origin.

LECTURE 4
MATH 261A

LECTURES BY: PROFESSOR DAVID NADLER
NOTES BY: JACKSON VAN DYKE

Office hours are officially Thursday 12:30-2 in Evans 740 or maybe Evans 814. Midterms will be short, potentially multiple choice. Then you can probably just drop the bad one. We are nearing the end of the basic intro part of the class, and will soon be moving to representation theory and structure theory, so there will likely be a midterm soon.

1. Sources of Lie algebras

Recall from last time, we defined Lie algebras, and we talked about where they come from. Recall the sources are:

(1) Whenever you have an associative algebra A, you can consider the derivations $\operatorname{Der}(A)$, and this is a Lie algebra.
(2) For A an associative algebra, we can just forget the fact that it's an algebra, and just remember the $[\cdot,\cdot]$ structure.

Example 1. The key example of the first one is the algebra $\mathcal{C}^\infty(M)$, and then $\operatorname{Vect}(M) = \operatorname{Der}(\mathcal{C}^\infty(M))$.

Example 2. The key example of the second is $A = \operatorname{Diff}(M)$ where we just think of this as a Lie algebra directly.

1.1. **Enveloping algebras.**

In the case of the examples above, $\operatorname{Vect}(M) \to \operatorname{Diff}(M)$. One might hope that the following is the case, though it is not.

Warning 1. $\operatorname{Diff}(M) \neq \mathcal{U}\operatorname{Vect}(M)$

The functor $\mathcal{U} : \textbf{Lie-Alg} \to k\textbf{-Alg}_{ass}$ is the adjoint functor to the forgetful functor. Explicitly, for $\mathfrak{g}$ a Lie algebra,

$$\mathcal{U}\mathfrak{g} = \bigoplus_{n=0}^{\infty} \mathfrak{g}^{\otimes n} / (x \otimes y - y \otimes x = [x,y])$$

Before modding out, this is just sums of words of the elements of $\mathfrak{g}$.

Remark 1. So why is the above warning true? Well $\mathfrak{g}^0$ is just k, but the zeroth portion of $\operatorname{Diff}(M)$ is smooth functions. So the corrected relationship is that the sheaf of differential operators is the universal enveloping algebroid of the tangent sheaf. The sheaf of differential operators is somehow a universal construction of this sheaf of vector fields.

Date: September 4, 2018.

2. Associating a Lie algebra to a Lie group

For G a Lie group, then T_eG, the tangent space at e is a Lie algebra.

Example 3 (Meta-example)**.** For $G = \mathrm{Diffeo}(M)$, the group of diffeomorphisms of M, what is $\mathfrak{g} = T_e \mathrm{Diffeo}(M)$? It is $\mathrm{Vect}(M)$. In any sense that one might conceive of, this consists of infinitesimal diffeomorphisms, or basically vector fields. G consists of the symmetries of something, and the identity is a god-given symmetry, and then we are looking for symmetries nearby. Open neighborhoods are already too complicated, so we just want to consider the linearization.

Example 4. Let $G = \mathrm{GL}(n, \mathbb{C}) \circlearrowright V = \mathbb{C}^n$. We won't use it, but it just so happens that it is acting on a vector space. So we have a map $G = \mathrm{GL}(n, \mathbb{C}) \xrightarrow{\sim} \mathrm{Aut}(V)$.

So now we want to look at the Lie algebra $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$ and understand what the bracket is all about. $\mathrm{GL}(n, \mathbb{C})$ is open inside all $n \times n$ matrices, $M(n, \mathbb{C})$ so the tangent space at any point is also just $M(n, \mathbb{C}) = \mathrm{End}(V)$.

Exercise 1. Show that the Lie algebra structure on $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$ is just the usual commutator $[A, B] = AB - BA$.

2.1. Vector fields.

Now we return to considering a general Lie group $G \circlearrowright X$ for some manifold X. Then we can differentiate this action, i.e. we have a map $\alpha : G \times X \to X$ and we can differentiate to get a map $T(\alpha) : TG \times TX \to TX$. Now we can restrict to $T_eG \times X \to TX$ where this copy of X is regarded as the zero-section of X.

Then we have a map of vector bundles $A : \mathfrak{g} \times X \to TX$ so this is a moment where we have used the fact that we are taking the tangent space at the identity in particular. Now we can pass to global sections, so for each $x \in X$, we obtain a linear map $A_x : \mathfrak{g} \to T_xX$, called the infinitesimal action map at $x \in X$.

All together, we obtain a linear map $\mathfrak{g} \to \mathrm{Vect}(X)$ which maps any vector field $v \mapsto \tilde{v}$ such that $\tilde{v}_x = A_x v$.

Example 5. If G consists of diffeomorphisms, then the tangent space at the identity consists of vector fields, so this construction gives us the identity.

The idea is that for $G \circlearrowright X$, we can say we are looking at a map of pairs $G \to \mathrm{Diffeo}(X)$, which is a group homomorphism, and from this, we saw $T_eG \to T_e \mathrm{Diffeo}(X) = \mathrm{Vect}(X)$, which will be a Lie algebra homomorphism when we understand the Lie structure on $\mathfrak{g}$. So basically the goal is to find a Lie algebra structure on $\mathfrak{g}$ such that this map is always a Lie algebra homomorphism.

Example 6. Consider $\mathrm{GL}(1, \mathbb{C}) = \mathbb{C}^\times \circlearrowright V = \mathbb{C}$. The action map is $\alpha : \mathbb{C}^\times \times \mathbb{C} \to \mathbb{C}$ which takes $\alpha(z, x) \to zx$. Now we want to unwind the definitions in order to see what this map $\mathfrak{g} \to \mathrm{Vect}(X)$ looks like in this case.

First of all, $\mathfrak{g} = \mathbb{C}$. Now we want to understand how to differentiate α. So for an element $v \in \mathfrak{g}$, we should get vector field on $\mathbb{C}$.

Remark 2. The general construction is as follows: Consider a map $F : M \to N$. If we have a point $x \in M$ and a vector $v \in T_xM$, and we want to somehow transport v to $T_{f(x)}N$, then we take an arbitrary path $\gamma : \mathbb{R} \to M$ such that the tangent line to γ at x is v, then we can map this to $F(\gamma)$, and take $F(\gamma)'(0)$, to get our $TF(v) \in T_{f(x)}N$.

Let's take $v = \partial_z$, and a path which has v as its tangent at 1. Take $\gamma(t) = e^t$, so this is a path from $\mathbb{R} \to G = \mathbb{C}^\times$, such that $\gamma(0) = e$, and $\gamma'(0) = (1,0) = \partial_z$. Acting by $\gamma(t)$ for small t gives a small motion of $\mathbb{C}$.

$$\alpha(\gamma(t), z) = \gamma(t) z = e^t z$$

So this is the image of the path, and we just need to differentiate with respect to t, and find $\tilde{v} = z\partial_z$.

Example 7. Consider $\mathrm{SL}(2,\mathbb{C}) \circlearrowright X = \mathbb{CP}^1$. The goal is again to calculate the map $\mathfrak{g} \to \mathrm{Vect}\left(\mathbb{CP}^1\right)$. For any vector field on $\mathbb{CP}^1$, we can restrict to the same vector field on $\mathbb{CP}^1 \setminus \infty$,

$$\begin{tikzcd}
\mathfrak{g} \arrow[r] \arrow[dr] & \mathrm{Vect}\left(\mathbb{CP}^1\right) \arrow[d] \\
& \mathrm{Vect}\left(\mathbb{CP}^1 \setminus \{\infty\}\right)
\end{tikzcd}$$

Recall $\mathrm{SL}(2,\mathbb{C}) = \{\det g = 1\}$. So this is sitting inside of $\mathbb{C}^4$, and the tangent space is

$$\mathfrak{sl}(2,\mathbb{C}) = \{x \mid \operatorname{tr} x = 0\} \ .$$

(The reason is, if we start out with the identity matrix, and want to add an extra matrix up to some multiple of ϵ, and maintain that the determinant is 0, we have:

$$\begin{aligned}
\det\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \epsilon \begin{pmatrix} 1 & b \\ c & d \end{pmatrix}\right) &= \det \begin{pmatrix} 1+\epsilon a & \epsilon b \\ \epsilon c & 1+\epsilon d \end{pmatrix} \\
&= (1+\epsilon a)(1-\epsilon d) - \epsilon^2 bc = 1 + \epsilon(a+d) + \epsilon^2 ad
\end{aligned}$$

so we just require that the additional matrices have trace 0.)

Now for each $x \in \mathfrak{sl}(2,\mathbb{C})$, we want some $f(s)\partial_s \in \mathrm{Vect}(\mathbb{C})$. So choose our favorite basis

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Now define $\gamma_M(t) = e^{tM}$ to get:

$$\gamma_H(t) = e^{tH} = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \quad \gamma_E(t) = e^{tE} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \quad \gamma_F(t) = e^{tF} = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$$

now we just apply these to our elements of $\mathbb{CP}^1$.

$$\gamma_F(t) \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ at+b \end{pmatrix}$$

so it took a point $s = b/a$ and transformed it into $s_t = (at+b)/a$. Now we take the derivative, and evaluate at $t = 0$ to get 1, so under this map

$$F \mapsto \partial_s \ .$$

Similarly, we can calculate

$$\gamma_H(t) \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} e^t a \\ e^{-t} b \end{pmatrix}$$

so $s = b/a \mapsto s_t = e^{-2t} b/a$, and again we differentiate, to discover that

$$H \mapsto -2s\partial_s \ .$$

Finally, we get

$$\gamma_E(t)\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a+bt \\ b \end{pmatrix}$$

so $s = b/a \mapsto b/(a+bt)$, and now differentiating, we get $-b^2/(a+bt)^2$, so finally

$$E \mapsto -s^2\partial_s \ .$$

Let's now consider $G \circlearrowright X = G$ acting by left multiplication. In this case, we get a map $\mathfrak{g} \to \mathrm{Vect}(G)$.

Let $H \circlearrowright Y$, then we can talk about $\mathrm{Vect}(Y)^H \subseteq \mathrm{Vect}(Y)$ the H-invariant vector fields.

Exercise 2. $\mathrm{Vect}(Y)^H \subseteq \mathrm{Vect}(Y)$ is a Lie subalgebra.

Lemma 1. *The image of this map is precisely the collection of right-invariant vector fields* $\mathrm{Vect}(G)^r$.

Proof. The claim here is that $\mathfrak{g} \xrightarrow{A} \mathrm{Vect}(G)^r$ is an isomorphism of Lie algebras. The fact that the image is contained in the right-invariant vector fields follows from commuting the right action with the left action.

The fact that it is an isomorphism follows from the fact that $A|_e = \mathrm{id} : \mathfrak{g} \to \mathfrak{g}$. $\square$

Definition 1. The Lie algebra structure on $\mathfrak{g}$ is transported via $\mathfrak{g} \xrightarrow{\sim} \mathrm{Vect}(G)^r \subset \mathrm{Vect}(G)$ which is a sub Lie-algebra.

Theorem 1. *If G acts on arbitrary X, then the map $\mathfrak{g} \to \mathrm{Vect}(X)$ is a Lie algebra homomorphism.*

LECTURE 5

LECTURES BY: PROFESSOR DAVID NADLER
NOTES BY: JACKSON VAN DYKE

The first midterm will be on Tuesday September 18.

1. Right-invariant vector fields

Recall that we have a functor

$$\textbf{Lie-Gp} \longrightarrow \textbf{Lie-Alg}$$

$$G \longrightarrow T_e G = \mathfrak{g} = \mathrm{Vect}\,(G)^r$$

where $\mathrm{Vect}\,(G)^r$ is the collection of right-invariant vector fields. This means for every morphism $h : H \to G$, we have a morphism $T\,(h) : \mathfrak{h} \to \mathfrak{g}$. A priori this is a map between these things, but potentially not a morphism of Lie algebras, though this will follow from the following.

We also have an association

$$\textbf{LieGroupActions} \rightsquigarrow \textbf{LieAlgActions}$$

where something on the left is of the form $G \to \mathrm{Diffeo}\,(X)$, which is then associated to $\mathfrak{g} \to \mathrm{Vect}\,(X)$.

Example 1. Consider the right-invariant vector fields on $\mathrm{GL}\,(n, \mathbb{C})$. This is an open subset of $\mathbb{R}^{n^2}$. How do we construct right-invariant vector fields? Well we looked at the left action of G on G, differentiated it, and then this gave us these vector fields.

Recall $T_e G = \mathfrak{g} = \mathfrak{gl}\,(n, \mathbb{C}) = M\,(n, \mathbb{C})$. Now let's take some tangent vector, and think about extending it to a right-invariant vector field. Recall we have an isomorphism:

Lemma 1. *In this example,*

$$\mathfrak{g} \xrightarrow{\sim} \mathrm{Vect}\,(G)^r$$

$$v \longmapsto \tilde{v}$$

Now pick a path $\gamma : \mathbb{R} \to G$, a one-parameter subgroup. This just means γ is a group homomorphism between $\mathbb{R}$ and G. For $G = \mathrm{GL}\,(n, \mathbb{C})$, take

$$\gamma\,(t) = e^{tv} = I + tv + \frac{t^2 v^2}{2} + \cdots$$

In general, for $v \in \mathfrak{g}$, we extend this to a right-invariant vector field $\tilde{v} \in \mathrm{Vect}\,(G)^r$. Now recall:

Date: September 6, 2018.

Theorem 1. *There exists unique local solutions to ODEs.*

In this situation, this means that given any vector field in a small enough ball, we can find a "motion" of the space that integrates this vector field for small enough time. I.e. vector fields integrate locally in space and time. Sometimes they integrate globally, but then you have to worry that you might "fall" off your space. So now look for an integral curve γ (check existence globally) of $\tilde{v}$ such that $\gamma(0) = e$.

Exercise 1. Check that γ extends uniquely to a 1-parameter subgroup. I.e. check that this construction not only gives us a map $(-\epsilon, \epsilon) \to G$, but in fact $\mathbb{R} \to G$ and that it is a homomorphism. [Hint: The fact that it's a homomorphism is almost obvious, and then you can use this fact to get the extension.]

So in the case of $\mathrm{GL}(n, \mathbb{C})$, this is the unique such 1-parameter subgroup. So now what we want to do is take some $n \times n$ matrix and get a right-invariant vector field from it. To construct $\tilde{v}$, consider γ acting on the left and differentiate with respect to t. Let $g \in \mathrm{GL}(n, \mathbb{C})$. Then we send

$$g \mapsto \gamma(t) \cdot g = e^{tv} g$$

Then differentiate to get

$$ve^{tv} g + e^{tv} g'|_{t=0} = vg$$

Note that v is an $n \times n$ matrix, which we want to picture as a tangent vector at the identity. So now the question is, if you stand at g, what is the $n \times n$ matrix which is telling you the value of $\tilde{v}$ at g, and the answer is vg. This all takes advantage of the fact that we are working in an ambient $\mathbb{R}^{n^2}$. In general, we can only write the following:

$$\tilde{v}|_g = R_g \cdot v$$

which is just the statement that it is right-invariant.

2. Right-invariant vs. left-invariant

Now we might wonder, why this is right invariant rather than left invariant. It's clear that G acting on the right gives a map from G to the G-equivariant automorphisms of G, $\mathrm{Aut}^G(G)$, and this map is also clearly injective. But any such automorphism is determined by the image of the identity, which allows us to show that this is in fact surjective as well.

In conclusion, the map $\mathfrak{g} \to \mathbf{Vect}(G)$ lands in the right invariant vector fields.

Exercise 2. Suppose this whole theory was developed with right actions instead of left actions. So we sent $v \to \tilde{v}^r$, and someone else sends $v \mapsto \tilde{v}^l$. Write a formula relating $\tilde{v}^r|_g$ and $\tilde{v}^l|_g$.

Solution. For $G = \mathrm{GL}(n, \mathbb{C})$ we just have $\tilde{v}^r|_g = vg$ and $\tilde{v}^l|_g = gv$. Then

$$\tilde{v}^l|_g = \widetilde{gvg^{-1}}^r|_g$$

So we just have to conjugate it.

Note also that $\tilde{v}^l|_\gamma = \tilde{v}^r|_\gamma$ since $\tilde{v}^l|_\gamma = \gamma \cdot v = e^{tv} \cdot v$ and $\tilde{v}^r|_\gamma = v \cdot \gamma = v \cdot e^{tv} = e^{tv} \cdot v$. So they certainly agree at the identity, but in general they will disagree elsewhere. The point here, is that the image of $\mathbb{R}$ in G is abelian, since $\mathbb{R}$ is abelian, so anywhere in this image, conjugation doesn't do anything, and they agree everywhere on γ.

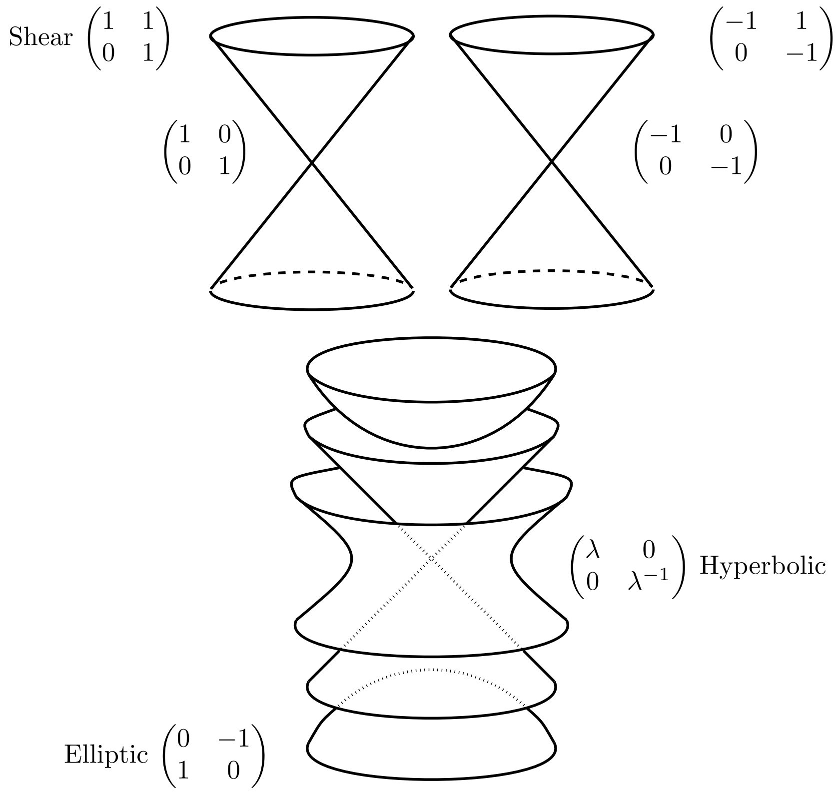

Figure 1. The cones which make up the orbits of the action of $\mathrm{SL}(2,\mathbb{R})$ on itself under conjugation. Note that for $\mathrm{SL}(2,\mathbb{C})$, the hyperboloid of two-sheets is not present since these matrices are then diagonalizable.

3. Conjugation and adjoint representations

We now consider conjugation in general as an additional natural action of G on itself.

Warning 1. Action by conjugation is neither free nor transitive.

Example 2. Consider the action of $\mathrm{GL}(n,\mathbb{C})$ on itself under conjugation. The orbits of this action are indexed by the so-called Jordan forms with nonzero eigenvalues.

For $G = \mathrm{SL}(2,\mathbb{C})$, the picture here is as in fig. 1. The stabilizer at $\mathbb{1}$ and $-\mathbb{1}$ is just G, and then

$$\mathrm{Stab}\left\langle \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\right\rangle = \left\langle \begin{pmatrix}1 & *\\ 0 & 1\end{pmatrix}\right\rangle \qquad \mathrm{Stab}\left\langle \begin{pmatrix}-1 & 1\\ 0 & -1\end{pmatrix}\right\rangle = \left\langle \begin{pmatrix}1 & *\\ 0 & 1\end{pmatrix}\right\rangle$$

Finally we have:

$$\mathrm{Stab}\left\langle \begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}\right\rangle = \left\langle \begin{pmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{pmatrix}\right\rangle$$

for $\alpha \in \mathbb{C}^\times$.

If we quotient by conjugation, then only the trace is well defined, because for any matrix, we either get λ or λ^{-1} under the map $\operatorname{tr} : G \to \mathbb{C}$. At any point we see $\lambda + \lambda^{-1}$, and then there are two special points ± 2, each of which somehow has these cones living above it, so they have a sort of fuzzy piece above it corresponding to the open piece. So this is some sort of strange non-Hausdorff space. The main takeaway is that the quotient looks like a line given by the trace.

We can see that the shear matrices form a cone as follows. Notice that they must have $\operatorname{Tr} A = 2$, and $\det A = 1$. So if we write the matrix as:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

this means $a + d = 2$ and $ad - bc = 1$, so $d(2-d) - bc = 1$. Now sending $d \mapsto d - 1$ we get $(d+1)(d-1) + bc = 1$ or

$$d^2 + bc = 0$$

now the quadratic form associated to this is

$$\begin{pmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

The eigenvalues of this matrix are $\pm 1/2$ and 1, so this corresponds to the equation

$$\frac{1}{2}x^2 - \frac{1}{2}y^2 + z^2 = 0$$

which is the equation for a cone.

Now, as usual, we want to differentiate this and see what sort of structures we get. So let's restrict $TG \times TG \to TG$ to $G \times T_eG \to T_eG$ where the first copy of TG has been restricted to the zero section. This is a homomorphism $\operatorname{Ad} : G \to \operatorname{GL}(\mathfrak{g})$, called the adjoint representation. All we've done here is take the conjugation action of G on itself, and then look at what this does to tangent vectors at the identity.

Example 3. For $G = \operatorname{GL}(n, \mathbb{C}) \circlearrowright M(n, \mathbb{C})$, the adjoint action is just $(g, h) \mapsto ghg^{-1}$, and we are differentiating with respect to h, and we get $(g, h') \mapsto gh'g^{-1}$. So the adjoint action of $\operatorname{GL}(n, \mathbb{C})$ is just given by this formula.

So the first thing we get from considering the conjugation, is we get a linearization of it, which is a conjugation of G on its Lie algebra. The following claim shows that it doesn't just act by any old matrices, but in fact by matrices which preserve the Lie algebra structure:

Claim 1. $G \to \operatorname{GL}(\mathfrak{g})$ acts by Lie algebra isomorphisms. I.e. $[\operatorname{Ad}_g v, \operatorname{Ad}_g w] = \operatorname{Ad}_g([v, w])$.

Proof. Recall the left multiplication action $L : G \times G \to G$. Now we can act on everything by conjugation:

$$\begin{tikzcd} (g,h) \arrow[r, mapsto, "L"] \arrow[d, mapsto] & gh \arrow[d, mapsto] \\ (kgk^{-1}, khk^{-1}) \arrow[r, mapsto, "L"] & kghk^{-1} \end{tikzcd}$$

Recall that $[v, w] = [\tilde{v}, \tilde{w}]\,|_e$. Now we can act by Ad_g on this expression to get:

$$\mathrm{Ad}_g\,[v, w] = \mathrm{Ad}_g\left([\tilde{v}, \tilde{w}]\,|_e\right) = \mathrm{Ad}_g\,[\tilde{v}, \tilde{w}]\,|_e = [\mathrm{Ad}_g\,\tilde{v}, \mathrm{Ad}_g\,\tilde{w}]\,|_e = \left[\widetilde{\mathrm{Ad}_g\,v}, \widetilde{\mathrm{Ad}_g\,w}\right]|_e$$

by right invariance. $\square$

Now we want to differentiate this with respect to the first variable as well. So instead restrict to $\mathrm{ad} : T_eG \times T_eG \to T_eG$, so this is a map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ which maps $(v, w) \mapsto \mathrm{ad}_v\,w$.

Theorem 2. $\mathrm{ad}_v\,(w) = [v, w]$

LECTURE 6
MATH 261A

LECTURES BY: PROFESSOR DAVID NADLER
NOTES BY: JACKSON VAN DYKE

The midterm will cover structure theory, and will be multiple choice. We will continue with geometric structure theory, and next time we will move on to representation theory of Lie algebras.

1. Adjoint representations

Recall that for every $g \in G$ we got the Ad_g map in the following way: We know $G \circlearrowright G$ by conjugation, and then consider the induced action $G \circlearrowright TG$, and in particular the action $G \circlearrowright T_e G \simeq \mathfrak{g}$ which we call Ad.

Write the left action $\alpha : G \times G \to G$. Now α is invariant under another G-action, in particular, the diagram

$$\begin{tikzcd} G \times (G \times G) \arrow[r, "\mathrm{id} \times \alpha"] \arrow[d, "\mathrm{conj.} \times \alpha"] & G \times G \arrow[d, "\alpha"] \\ G \times G \arrow[r, "\alpha"] & G \end{tikzcd}$$

commutes. The point is, the vertical arrows are an additional invariance.

Lemma 1. *For $g \in G$, $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$ is a Lie algebra morphism. This means that*

$$[\mathrm{Ad}_g v, \mathrm{Ad}_g w] = \mathrm{Ad}_g([v, w])$$

Proof. Consider $\mathrm{Ad}_g([v, w])$. We can rewrite this as:

$$\begin{aligned} \mathrm{Ad}_g([v, w]) &= \mathrm{Ad}_g([\tilde{v}, \tilde{w}]_e) \\ &= ([\mathrm{Ad}_g \tilde{v}, \mathrm{Ad}_g \tilde{w}] \,|_e) \\ &= \left(\widetilde{\mathrm{Ad}_g(v)}, \widetilde{\mathrm{Ad}_g(w)}\right) |_e \end{aligned}$$

where we have used the fact that Ad_g is a diffeomorphism to get to the second line. □

Recall that if we differentiate again, only this time wrt the first G, we get $\mathrm{ad} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. More formally,

$$\mathrm{ad}_v(w) = \frac{d}{dt}\left(\mathrm{Ad}_{\gamma(t)}(w)\right) |_{t=0}$$

where γ is some path such that $\gamma(0) = e$ and $\gamma'(0) = v$. Then we have the following theorem:

Theorem 1. $\mathrm{ad}_v(w) = [v, w]$

Date: September 11, 2018.

Proof. We know

$$[v, w] = [\tilde{v}, \tilde{w}]\,|_e = \left(\frac{d}{dt} L_{\gamma(t)}(\tilde{w})\right)|_{t=0}|_e$$

and since $\tilde{w}$ is right-invariant, we are done. □

2. Geometric structure theory

Assume everything is finite dimensional. Recall that we have the functor:

$$\textbf{Lie-Gp} \longrightarrow \textbf{Lie-Alg}$$

$$G \longrightarrow T_e G = \mathfrak{g}$$

Then we have the following:

Theorem 2. *This functor is an equivalence when restricted to connected, simply-connected Lie groups.*

Example 1. Consider $\mathfrak{g}$ abelian, so it is isomorphic to $\mathbb{R}^n$ (or just $\mathbb{C}^n$.) Since $\mathfrak{g}$ is abelian, $[\cdot, \cdot]$ is just 0. Which Lie groups have this algebra? The theorem tells us there is a unique simply connected one, namely $\mathbb{R}^n$ with addition. But then there is a whole tower of things covered by this such as $\left(S^1\right)^n$, $\left(\mathbb{C}^\times\right)^n$, $\left(\mathbb{C}^n/\mathbb{Z}^{2n}\right)$ and many more.

Example 2. Consider $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$. Let's come up all the potential Lie groups which give rise to this algebra. Of course $\mathrm{SL}(2, \mathbb{C})$ gives rise to this, but this might not be the unique one we are looking for if it is not simply-connected, so we are instead looking for the universal cover of this.

What is the fundamental group of $\mathrm{SL}(2, \mathbb{C})$? We know $\mathrm{SL}(2, \mathbb{C}) \circlearrowright \mathbb{C}^2$. This has two orbits, i.e. when $v = 0$ and $v \neq 0$. The stabilizer of the first is everything, and

$$\mathrm{Stab}\begin{pmatrix}1\\0\end{pmatrix} = \left\langle \begin{pmatrix}1 & u\\0 & 1\end{pmatrix}\right\rangle$$

This means

$$\mathrm{SL}(2, \mathbb{C}) / \mathrm{Stab} \simeq \mathrm{SL}(2, \mathbb{C}) / \mathbb{C} \simeq \mathbb{C}^2 \setminus \{0\}$$

This means $\pi_1(\mathrm{SL}(2, \mathbb{C}))$ is the same as π_1 of the complement of 0 in $\mathbb{C}^2$. But this is homotopy equivalent to S^3, which is simply connected (it has trivial π_1) so the unique simply connected Lie group is just $\mathrm{SL}(2, \mathbb{C})$.

But what other Lie groups might give rise to this algebra? This is really just considering things that $\mathrm{SL}(2, \mathbb{C})$ covers. If we consider the center

$$Z = Z(\mathrm{SL}(2, \mathbb{C})) \simeq \mathbb{Z}/2 = \left\langle \begin{pmatrix}\pm 1 & 0\\0 & \pm 1\end{pmatrix}\right\rangle$$

and mod out by this, we get $\mathrm{PSL}(2, \mathbb{C}) = \mathrm{SL}(2, \mathbb{C}) / Z$ which has $\pi_1 \simeq \mathbb{Z}/2$. These turn out to be the only two. Equivalently, $\langle 1 \rangle$ and $Z = \mathbb{Z}/2$ are the only two discrete normal subgroups of $\mathrm{SL}(2, \mathbb{C})$.

Exercise 1. Show $\mathrm{PSL}(2, \mathbb{C}) = \mathrm{SO}(3, \mathbb{C})$

Solution. Recall $\mathrm{SO}(3)$ consists of M such that $M^T = M^{-1}$ and $\det M = 1$. We want to send any A and $-A$ to the same $B \in \mathrm{SO}(3)$.

Recall $\mathrm{Ad} : G \times \mathfrak{g} \to \mathfrak{g}$, so for $g \in G$, $\mathrm{Ad}_g \in \mathrm{GL}(\mathfrak{g})$. So apply this to $G = \mathrm{SL}(2,\mathbb{C})$. Given $A \in \mathrm{SL}(2,\mathbb{C})$ and $v \in \mathfrak{sl}(2,\mathbb{C})$, we send this to $AvA^{-1} \in \mathfrak{sl}(2,\mathbb{C})$, and $\mathfrak{sl}(2,\mathbb{C})$ is 3-dimensional, so this makes Ad_A into a 3×3 matrix. So the question is, what 3×3 matrices do we obtain? To find out, we consider the inner product $\langle v, w\rangle = \mathrm{Tr}(vw)$. So if

$$v = \begin{pmatrix} x & y \\ z & -x \end{pmatrix} \qquad\qquad w = \begin{pmatrix} r & s \\ t & -r \end{pmatrix}$$

the inner product is

$$\langle v, w\rangle = xr + yt + zs + xr$$

and in particular,

$$\langle v, v\rangle = x^2 + yz + zy + x^2 = 2\left(x^2 + yz\right)$$

which is a nondegenerate quadratic form. But there is only one nondegenerate quadratic form on a complex vector space, i.e. in a different basis, this is just the sum of the squares. Now this inner product is clearly invariant under the $\mathrm{SL}(2,\mathbb{C})$ action, since

$$\mathrm{Tr}\left(AvA^{-1}AwA^{-1}\right) = \mathrm{Tr}\left(AvwA^{-1}\right) = \mathrm{Tr}(vw)$$

so the matrices preserve this quadratic form. So these 3×3 matrices land in the orthogonal group of this quadratic form, now we just have to check it has determinant 1. To do this, we consider the following basis for $\mathfrak{sl}(2,\mathbb{C})$:

$$v_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad v_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Now we calculate the action of some arbitrary $A \in \mathrm{SL}(2,\mathbb{C})$ as

$$\begin{aligned} Av_1A^{-1} &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \\ &= \begin{pmatrix} ad+bc & -2ab \\ 2dc & -(ad+bc) \end{pmatrix} \end{aligned}$$

Completing the same calculation for the other basis elements, we can express Ad_A as the following matrix:

$$\begin{pmatrix} ad+bc & -ac & bd \\ -2ab & a^2 & -b^2 \\ 2dc & -c^2 & d^2 \end{pmatrix} \tag{1}$$

then we can calculate:

$$\begin{aligned} &(ad+bc)\left(a^2d^2 - b^2c^2\right) + ac\left(-2abd^2 + 2b^2cd\right) + bd\left(2abc^2 - 2a^2cd\right) \\ &= a^3d^3 - ab^2c^2d + a^2bcd^2 - b^3c^3 - 2a^2bcd^2 + 2ab^2c^2d + 2ab^2c^2d - 2a^2bcd^2 \\ &= (ad-bc)^3 = 1 \end{aligned} \tag{2}$$

since $A \in \mathrm{SL}(2,\mathbb{C})$.

This means the image lands in $\mathrm{SO}(3,\mathbb{C})$, where this is the orthogonal group with respect to the above inner product, which is fine since all such inner products are the same. So now we just need to show the kernel of this map is the center. But from the matrix in (1) we can see this directly, since if A is in the kernel, $a = d = \pm 1$ and therefore $c = d = 0$ as desired.

Example 3. What about $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$. If we try the same game as in the complex case, we find that $\mathrm{SL}(2, \mathbb{R}) / \mathrm{Stab} \simeq \mathbb{R}^2 \setminus \{0\}$ is the orbit, which is homotopy equivalent to S^1, which has $\pi_1 = \mathbb{Z}$. The universal cover, $\widetilde{\mathrm{SL}(2, \mathbb{R})}$, has a map to $\mathrm{SL}(2, \mathbb{R})$ with fibers $\mathbb{Z}$.

Exercise 2. If we have a group which is not simply-connected, then the universal cover is naturally a Lie group.

Solution. The universal cover is space of homotopy classes of paths from a base point. Then we can multiply two paths pointwise to get a group, and the projection is homomorphism.

Warning 1. This universal cover does not have any finite dimensional representations, so it cannot be viewed as consisting of matrices.

We have just been assuming this so far, but for $G = \mathrm{GL}(n, \mathbb{C})$, the fact that $\mathrm{ad}_v(w) = [v, w]$ means that $[\tilde{v}, \tilde{w}] = vw - wv$, so the bracket on $\mathfrak{gl}(n, \mathbb{C})$ is truly the commutator of the matrices since

$$\frac{d}{dt}\left(\mathrm{Ad}_{\gamma(t)}\, w\gamma(t)^{-1}\right) = vw + w(-v)$$

Theorem 3 (Ado)**.** *Any finite dimensional Lie algebra is a subalgebra of $\mathfrak{gl}(n, \mathbb{R})$ for some n.*

Partial proof. Assume $Z(\mathfrak{g}) = \langle 0 \rangle$, so nothing has bracket 0 with everything. We know $\mathrm{ad} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, which we can view as a map $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}) = \mathfrak{gl}(n, \mathbb{R})$ where $n = \dim \mathfrak{g}$. Since the center is trivial, the kernel is trivial, so this is an injection. □

2.1. **Killing form.** The Killing form is an inner product $\mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ (or $\mathbb{R}$) where we take

$$\langle v, w \rangle_k = \mathrm{Tr}(\mathrm{ad}_v\, \mathrm{ad}_w)$$

which is a bilinear pairing.

Warning 2. This is not always nondegenerate.

For example, if $\mathfrak{g}$ is abelian, ad_v and ad_w are 0. Note that all of my matrices preserve this inner product. Now write $Q_k(v) = \langle v, v \rangle$, and then $\mathrm{Ad} : G \to \mathrm{O}(Q_k)$ and $\mathrm{ad} : \mathfrak{g} \to \mathfrak{o}(Q_k)$.

LECTURE 7
MATH 261A

LECTURES BY: PROFESSOR DAVID NADLER
NOTES BY: JACKSON VAN DYKE

Recall last time we were about to prove:[1]

Theorem 1. *The following functor*

$$\textbf{Lie-Gp} \longrightarrow \textbf{Lie-Alg}$$

$$G \longmapsto \mathfrak{g}$$

is an equivalence when restricted to connected, simply-connected groups.

Why is this a functor? I.e. why does $\varphi : H \to G$ induce a Lie-algebra homomorphism $d\varphi : \mathfrak{h} \to \mathfrak{g}$. Consider $H \circlearrowright G$ on the left via φ. Then

$$\begin{tikzcd} \mathrm{Vect}^r(H) \arrow[r] \arrow[dr, dashed] & \mathrm{Vect}(G) \\ & \mathrm{Vect}^r(G) \arrow[u, hook] \end{tikzcd}$$

Definition 1. Let H, G be Lie groups. Then they are said to be locally isomorphic if there is some neighborhood $U_H \subset G$ of $e \in H$ and some neighborhood $U_G \subset G$ of $e \in G$ and a diffeomorphism $\varphi : U_H \to U_G$ mapping $e \mapsto e$ such that for any $h_1, h_2 \in U_H$, $h_1 h_2 \in U_H$ iff $\varphi(h_1)\varphi(h_2) \in U_G$ and in this case,

$$\varphi(h_1, h_2) = \varphi(h_1)\varphi(h_2)$$

1. Examples

We will consider

Example 1. First of all $\mathbb{C}^n$ is the universal cover of $(\mathbb{C}^\times)^n = \mathbb{C}^n/\mathbb{Z}^n$ and so they are locally isomorphic.

Define $\mathrm{Spin}(n,\mathbb{C})$ to be the double cover of $\mathrm{SO}(n,\mathbb{C})$. For $n > 2$, $\mathrm{Spin}(n,\mathbb{C})$ is simply connected, so it is also the universal cover of $\mathrm{SO}(n,\mathbb{C})$.

Example 2. $\mathrm{SO}(1,\mathbb{C})$ is a single point, so $\mathrm{Spin}(1) \cong \mathbb{Z}/2\mathbb{Z}$.

Example 3. $\mathrm{SO}(2,\mathbb{C}) \cong \mathbb{C}^\times$, so $\mathrm{Spin}(2) = \mathbb{C}^\times$ as a double cover.

Example 4. $\mathrm{Spin}(3,\mathbb{C}) = \mathrm{SL}(2,\mathbb{C})$ and $\mathrm{SO}(3,\mathbb{C})$ are locally isomorphic. Recall we have the short exact sequence

$$\mathbb{Z}/2 \to \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SO}(3,\mathbb{C})$$

Date: September 13, 2018.
[1] According to professor Nadler, we will at least prove this by December...

Example 5. We seek to show that

$$\mathrm{Spin}(4,\mathbb{C}) = \widetilde{\mathrm{SO}}(4,\mathbb{C}) = \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$$

The first thing to find is $\dim \mathrm{SO}(n,\mathbb{C})$, which is of course $\dim \mathfrak{so}(n,\mathbb{C})$. Now differentiating $AA^T = I$ we get

$$X\left(A^T|_{t=0}\right) + AX^T|_{t=0} = 0$$

so $X + X^T = 0$. This means the dimension of this is $n(n-1)/2$, so it makes sense that $\mathrm{Spin}(4,\mathbb{C}) = \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$ To see this identification, we can consider $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}) \circlearrowright M_{2\times 2}(\mathbb{C}) \simeq \mathbb{C}^4$. Then we can take $Q = \det : M_{2\times 2}(\mathbb{C}) \to \mathbb{C}$ to be our quadratic form.

Notice that $\widetilde{\mathrm{SO}(4,\mathbb{C})}$ is a $\mathbb{Z}/2$ cover of $\mathrm{SL}(4,\mathbb{C})$, but this still has nontrivial center, so this is a $\mathbb{Z}/2$ cover of some sort of $PSO(4,\mathbb{C})$ which turns out to be. $\mathrm{SO}(3,\mathbb{C}) \times \mathrm{SO}(3,\mathbb{C})$ As it turns out, we can do the other possible $\mathbb{Z}/2$ quotients to get the diagram:

$$\begin{tikzcd}
 & \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}) \arrow[dl] \arrow[d] \arrow[dr] & \\
\mathrm{SO}(3,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}) \arrow[dr] & \mathrm{SO}(4,\mathbb{C}) \arrow[d] & \mathrm{SO}(3,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}) \arrow[dl] \\
 & \mathrm{SO}(3,\mathbb{C}) \times \mathrm{SO}(3,\mathbb{C}) &
\end{tikzcd}$$

This is the Galois covering diagram for the Galois group $\mathbb{Z}/2$.

Example 6. For $n = 5$ we have $\mathrm{Spin}(5,\mathbb{C}) = \mathrm{Sp}(4,\mathbb{C})$ and the diagram is just:

$$\begin{array}{c} \mathrm{Sp}(4) \\ \downarrow \\ \mathrm{SO}(5,\mathbb{C}) \end{array}$$

Example 7. For $n = 6$ we have $\mathrm{Spin}(6,\mathbb{C}) = \mathrm{SL}(4,\mathbb{C})$. This has the diagram:

$$\begin{array}{c} \mathrm{SL}(4) \\ \downarrow \\ \mathrm{SO}(6) \\ \downarrow \\ PSO(6) = \mathrm{PSL}(4,\mathbb{C}) \end{array}$$

where the arrows represent quotienting by $\mathbb{Z}/2$, even though $Z(\mathrm{SL}(4)) = \mathbb{Z}/4$. So we are quotienting by subgroups of the center to move down this tower.

This is the end of the spin group coincidences.

2. Lie's fundamental theorems

Theorem 2. *For H and G locally isomorphic, $\mathfrak{h}$ and $\mathfrak{g}$ are locally isomorphic.*

Theorem 3. *If $\mathfrak{g}$ and $\mathfrak{h}$ are locally isomorphic, then any Lie groups G and H which give rise to them are locally isomorphic, so they have the same universal cover.*

Theorem 4. *Every $\mathfrak{g}$ is the Lie algebra of some G.*

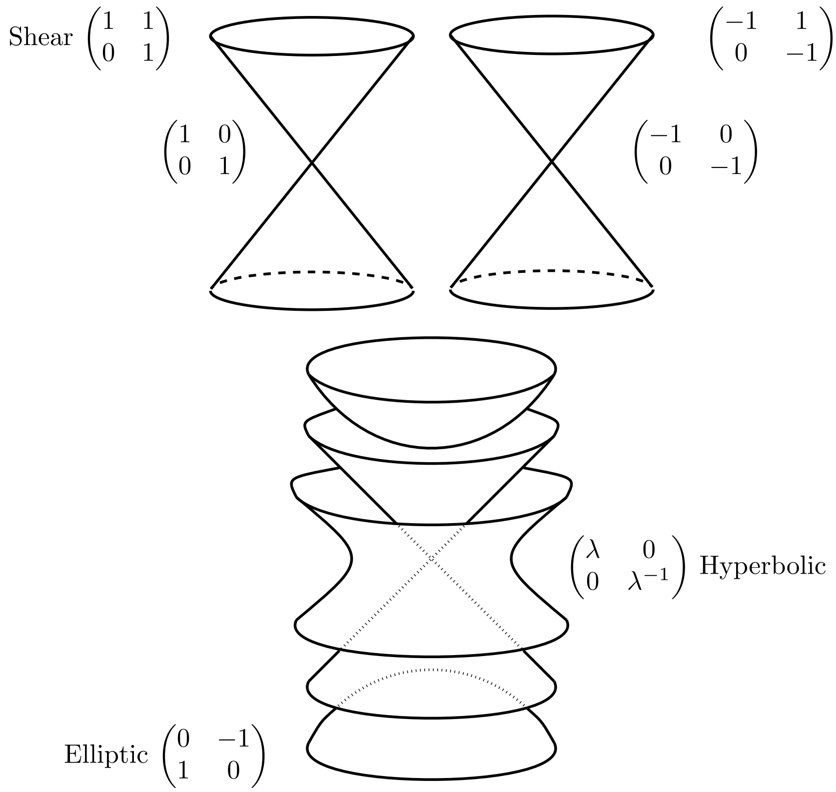

FIGURE 1. The cones which make up the orbits of the action of $\mathrm{SL}(2,\mathbb{R})$ on itself under conjugation. Note that for $\mathrm{SL}(2,\mathbb{C})$, the hyperboloid of two-sheets is not present since these matrices are then diagonalizable.

Proof. Recall Ado's theorem says that any finite dimensional Lie algebra is a subalgebra of $\mathfrak{gl}(n,\mathbb{R})$ for some n. I.e. it has a faithful representation. Note that if $Z(\mathfrak{g}) = \langle 0\rangle$, then $\mathrm{ad} : \mathfrak{g} \hookrightarrow \mathrm{GL}(\mathfrak{g})$

Take $G \subseteq \mathrm{GL}(n,\mathbb{R})$ to be generated by all 1-parameter subgroups generated by $\gamma(t) : \mathbb{R} \to G$ with $\gamma'(0) \in \mathfrak{g} \subseteq \mathfrak{gl}(n,\mathbb{R})$.

Then there are lots of things to check.

Exercise 1. Not every element of, for example $\mathrm{SL}(2,\mathbb{R})$, is in the image of some 1-parameter subgroup.

Recall this cone picture from fig. 1 Recall $\gamma(t) = e^{tv}$, then we can take $v \in \mathfrak{sl}(2,\mathbb{R})$ and put it in Jordan form so we get matrices of the types:

$$\begin{pmatrix}0 & 0\\0 & 0\end{pmatrix} \qquad \begin{pmatrix}a & 0\\0 & -a\end{pmatrix} \qquad \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix} \qquad \begin{pmatrix}0 & -b\\b & 0\end{pmatrix}$$

for $a \neq -a$ and $b \neq 0$. Now we go and write the exponentials of these things, and we get for example

$$\exp t \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} ct & -st \\ st & ct \end{pmatrix}$$

so we can get the negative identity, but we can't get the negative shears or negative hyperbolic elements.

Anyway, this is a sketch of a proof of the third theorem, or the essential surjectivity of the theorem from the beginning. $\square$

For any Lie group G, we have the exponential map $\exp : \mathfrak{g} \to G$ defined as the map such that if $\exp(v) = \gamma(1)$ where $\gamma(T) : \mathbb{R} \to G$, then $\gamma(0) = v$.

Note for $G \subseteq \mathrm{GL}(n, \mathbb{R})$, exp is the exponential we already know.

Exercise 2. Take the differential $T(\exp) : T\mathfrak{g} \to TG$, and restrict this to $\{0\} \times \mathfrak{g}$, which gives us a map $\mathfrak{g} \to T_e G$. Show this is the identity.

Lemma 1. *The image of one-parameter subgroups contains an open neighborhood of e.*

Proof. By the exercise, $\exp : \mathfrak{g} \to G$ is a local diffeomorphism from a neighborhood of 0 to a neighborhood of the identity. $\square$

Proof of the theorem. It remains to show the bijection on maps. We first show it is surjective. Consider a Lie algebra map $\varphi : \mathfrak{h} \to \mathfrak{g}$ and then we want a map $H \to G$ assuming H is simply connected.

Now we don't want to construct the actual map, but rather the graph of the map. We know a lot about subgroups, so we want to embed this problem in the context of constructing subgroups.

Consider the graph of φ as $\Gamma_\varphi \subseteq \mathfrak{h} \times \mathfrak{g}$. Since φ is a homomorphism, we can check that Γ_φ is a subalgebra, and now we can generate a subgroup of $H \times G$ whose Lie algebra will be this graph. Then this subgroup will be the graph of the desired map of Lie groups.

Then this is not a cover, since G is simply connected, so it's really a map, not a correspondence. $\square$

LECTURE 8
MATH 261A

LECTURES BY: PROFESSOR DAVID NADLER
NOTES BY: JACKSON VAN DYKE

The next midterm will probably be take home.

1. The last midterm question

Recall if we have a Lie group acting $G \circlearrowright X$ we get an infinitesimal action, which is a map $\mathfrak{g} \to \mathrm{Vect}(X)$ which is a map of Lie algebras, so it is linear. The moment map is effectively the transpose to this map:

$$\mu : T^*X \to \mathfrak{g}^*$$

which is somehow no more or less information than $\mathfrak{g} \to \mathrm{Vect}(X)$. Explicitly, for $x \in X$ and $\xi \in T_x^*X$,

$$\mu(x,\xi)(v) = \xi(\widetilde{v}_x)$$

for $v \in \mathfrak{g}$.

Exercise 1. We know $d\mu$ is a $\mathfrak{g}^*$ valued 1-form on T^*X. Then $\omega^{-1}(d\mu)$ is now a $\mathfrak{g}^*$-valued vector field on T^*X, and now this can be evaluated at $v \in \mathfrak{g}$, so we get $\widehat{v} = \omega^{-1}(d\mu)(v)$ which is now a vector field on T^*X. Show that this vector field is tangent to the zero-section, and gives us $\tilde{v}$. I.e. show $\widehat{v}|_X = \widetilde{v}$. This is somehow recovering the infinitesimal action from the moment map and symplectic structure.

So now we want to calculate this explicitly in the examples from the midterm.

Example 1. Let $\mathrm{GL}(1,\mathbb{R}) = \mathbb{R}^\times \circlearrowright \mathbb{R}$ by $r \cdot x = rx$. This action generates the vector field $\tilde{v} = x\partial_x$, so $\mu(x,\xi) = \xi(x\partial_x) = x\xi$.

Example 2. Let $\mathrm{GL}(1,\mathbb{R}) = \mathbb{R}^\times \circlearrowright \mathbb{R}^2$ by $r \cdot (x_1,x_2) = \left(rx_1, r^{-1}x_2\right)$. Then the vector field is $x_1\partial_{x_1} - x_2\partial_{x_2}$. The moment map is just $\mu = x_1\xi_1 - x_2\xi_2$.

Example 3. Now let $G \circlearrowright X = G$. In this case $T^*X = T^*G$ is parallelizable, so $T^*G = G \times \mathfrak{g}$, since $G \times \mathfrak{g} \xrightarrow{\sim} TG$ is just right-invariant vector fields, so it's just mapping $(g,v) \mapsto (g,\tilde{v}_g)$. This means function $\mu : T^*G \to \mathfrak{g}^*$ are just functions $G \times \mathfrak{g}^* \to \mathfrak{g}^*$.

If the action is trivial, the vector field is 0. This means $\mu = 0$.

If the action is left multiplication, then $\mu_l(g,\xi) = \xi$.

If the action is right multiplication, then

$$\mu_r(g,\xi)(v) = \mu_l(g,\xi)(\mathrm{Ad}_g v) = \mathrm{Ad}_g(\xi)(v)$$

Date: September 20, 2018.

2. Lie algebras

2.1. **Ignoring groups.** We have now developed enough theory to see that the theory of simply connected Lie groups is the same theory as finite dimensional lie algebras. Therefore we will now ignore Lie groups and focus on Lie algebras. Not because we don't care about them, but because we understand they are equivalent.

Professor Nadler doesn't know how to answer the following:

Exercise 2. For $\mathfrak{g}$ a Lie algebra, then we can associate it to G a connected, simply connected Lie group. What is the center of G?

Solution. This solution doesn't make sense until after lecture 13 at the earliest. We claim the following:

Claim 1. The center of the simply connected compact group G associated to a Lie algebra $\mathfrak{g}$ can be identified with the dual of the finite group $\Lambda/\mathbb{Z}R$ where Λ is the weight lattice and $\mathbb{Z}R$ is the root lattice.

If $\Lambda = \operatorname{Hom}(H, \mathbb{C}^\times)$ is the weight lattice and $\mathbb{Z}R$ is the root lattice, which is given by $\mathbb{Z}$-linear combinations of the nonzero eigenvalues of the adjoint representation ad, then we can write down the dual of these things to get:

$$\Lambda^* = \{X \in \mathfrak{h} \mid \forall L \in \Lambda, LX \in \mathbb{Z}\}$$
$$(\mathbb{Z}R)^* = \{X \in \mathfrak{h} \mid \forall \alpha \in \mathbb{Z}R, \alpha X \in \mathbb{Z}\}$$

now under the exponential map, $(\mathbb{Z}R)^*$ maps onto the center of H, which is the center of G, so we just need to quotient out by the kernel of the exponential, but this is exactly Λ^*.

Recall this is important because if $G \to G/\Gamma$ is some covering, then $\Gamma \subseteq Z(G)$. So knowing the center lets us calculate the types of covers and therefore all of the groups G which might give rise to $\mathfrak{g}$.

2.2. **Fields.** From now on we will focus on representation theory of Lie algebras. We can consider Lie algebras over any field.[1] We will usually let this be $\mathbb{C}$, but first we make some comments about the general setting. For any $\mathfrak{g}/k$, we can pass to $\mathfrak{g} \otimes_k \overline{k}/\overline{k}$ where we have extended all of the bracket operations linearly.

First note that in general this operation somehow loses information. That is, many different $\mathfrak{g}/k$ might go to the same $\mathfrak{g} \otimes_k \overline{k}/\overline{k}$.

Example 4. Consider $\mathfrak{sl}(2, \mathbb{R}) \to \mathfrak{sl}(2, \mathbb{C})$, and $\mathfrak{so}(3, \mathbb{R}) \to \mathfrak{so}(3, \mathbb{C})$. We already saw that $\mathfrak{sl}(2, \mathbb{C}) \simeq \mathfrak{so}(3, \mathbb{C})$. But the point is that $\mathfrak{sl}(2, \mathbb{R}) \not\simeq \mathfrak{so}(3, \mathbb{R})$. One way to see this, is that the universal cover of $\mathrm{SL}(2, \mathbb{R})$ is contractible and noncompact. Whereas the universal cover of $\mathrm{SO}(3, \mathbb{R})$ is $\mathrm{Spin}(3)$, which is compact.

In what follows, we will start with $\mathfrak{g}/\mathbb{C}$ finite dimensional. If time permits, one thing we could do is talk about what happens when k is not algebraically closed.

3. Rough classification

We won't worry too much about the details of these definitions or their relationship right now. We're more worried about getting a rough idea of what we are looking at.[2] Recall Ado's theorem, which says that $\mathfrak{g} \hookrightarrow \mathfrak{gl}(n, \mathbb{C})$ for some n. The

[1] Or rings, but we won't worry too much about this.

[2] Professor Nadler compared this to going to the Zoo. It is nice to have some idea what the big cat house is and what the reptile house is.

point being, that these always somehow come as matrices. This will be a theme throughout.

3.1. **Abelian Lie algebras.** First we might study $\mathfrak{g}$ abelian, so $[v,w]=0$ for all $v,w\in\mathfrak{g}$. Therefore these are just complex vector spaces of some finite dimension.

Example 5. The classic example of this is just diagonal matrices $\mathbb{C}^n\subseteq\mathfrak{gl}(n,\mathbb{C})$.

For arbitrary $\mathfrak{g}$, we can always associate a certain abelian Lie algebra to $\mathfrak{g}$, called its center which is defined as

$$\{v\in\mathfrak{g}\,|\,\forall w\in\mathfrak{g},[v,w]=0\}$$

Example 6. If we consider $\mathfrak{g}=\mathfrak{gl}(n,\mathbb{C})$, then the center is $\mathfrak{z}(\mathfrak{g})=\{zI_n\,|\,z\in\mathbb{C}\}$.

3.2. **Nilpotent Lie algebras.** For any $\mathfrak{g}$ we can define the commutator subalgebra $[\mathfrak{g},\mathfrak{g}]$ which consists of all linear combinations of commutators of elements of $\mathfrak{g}$. Then we can continue to take the commutator of this object with $\mathfrak{g}$ to get a series:

$$[[\mathfrak{g},\mathfrak{g}],\mathfrak{g}] \qquad [[[\mathfrak{g},\mathfrak{g}],\mathfrak{g}],\mathfrak{g}] \qquad \cdots$$

If this process ever reaches 0, we say $\mathfrak{g}$ is nilpotent.

Example 7. The classic example is strictly upper triangular $n\times n$ matrices written $\mathfrak{n}(n,\mathbb{C})$. If we take the commutator, we lose the super diagonal, and then each commutator after that we lose another diagonal.

Theorem 1. *If $\mathfrak{g}$ is nilpotent, then $\mathfrak{g}\subseteq\mathfrak{n}(n,\mathbb{C})$ for some n.*

Fact 1. *All subalgebras of nilpotent Lie algebras are nilpotent.*

3.3. **Solvable Lie algebras.** There are many equivalent definitions for solvable Lie algebras, but we define this to be a Lie algebra $\mathfrak{g}$ such that $[\mathfrak{g},\mathfrak{g}]$ is nilpotent. This doesn't mean $\mathfrak{g}$ is nilpotent, since this condition just says that:

$$[[\mathfrak{g},\mathfrak{g}],[\mathfrak{g},\mathfrak{g}]] \qquad [[[\mathfrak{g},\mathfrak{g}],[\mathfrak{g},\mathfrak{g}]],[\mathfrak{g},\mathfrak{g}]] \qquad \cdots$$

eventually reaches 0.

Example 8. The classic example of a solvable Lie algebra is $\mathfrak{b}(n,\mathbb{C})$ consisting of upper triangular matrices. Note that the commutator subalgebra $[\mathfrak{b},\mathfrak{b}]$ of course yields strictly upper triangular matrices $\mathfrak{n}$, which we already saw were nilpotent.

Theorem 2. *Any solvable Lie algebra $\mathfrak{g}$ is contained $\mathfrak{g}\subseteq\mathfrak{b}(n,\mathbb{C})$ for some n.*

3.4. **Simple.** There are many formulations of simple Lie algebras, but one is that $\mathfrak{g}$ is not abelian, and has no proper non-zero ideals. The non-abelian condition is basically just to omit $\mathbb{C}$.

Example 9. The classic example is $\mathfrak{sl}(n,\mathbb{C})$. Note that $\mathfrak{gl}(n,\mathbb{C})$ is not simple, since this looks like $\mathfrak{sl}(n,\mathbb{C})\oplus\mathbb{C}$, and therefore has two non-zero proper ideals.

Theorem 3. *If $\mathfrak{g}$ is simple, it somehow sits inside $\mathfrak{sl}(n,\mathbb{C})$.*

3.5. **Semisimple.** $\mathfrak{g}$ is semi-simple if it is a direct sum of simple Lie algebras.

Example 10. The classic example is

$$\bigoplus_i \mathfrak{sl}(n_i, \mathbb{C}) \qquad \sum_i n_i = n$$

so this is just sort of $n \times n$ block diagonal matrices where each block has trace 0.

Fact 2. $\mathfrak{g}$ *is semi-simple iff the* radical[3]*, which is the maximal solvable ideal,* $\mathfrak{rad}(\mathfrak{g}) = \langle 0 \rangle$.

Note that this also means semisimple Lie algebras have no center.

3.6. **Reductive.** The idea here is that $\mathfrak{g}$ is reductive if it is the direct sum of a semi-simple Lie algebra and an abelian Lie algebra:

$$\mathfrak{g} = \mathfrak{g}^{ss} \oplus \mathfrak{z} \ .$$

This abelian Lie algebra will of course also be the center of $\mathfrak{g}$.

Fact 3. $\mathfrak{g}$ *is reductive iff the radical* $\mathfrak{rad}(\mathfrak{g}) = \mathfrak{z}$ *is just the center.*

Example 11. A classic example is $\mathfrak{gl}(n, \mathbb{C})$. In some sense the reason we define this is, well, to include this, and also to contain

$$\bigoplus_{i=1}^{l} \mathfrak{gl}(n_i, \mathbb{C})$$

which consists of block matrices with no conditions on the blocks

Proposition 1. *This contains abelian Lie algebras as well as semi-simple Lie algebras.*

Fact 4. *The sum of any two nilpotent ideals is a nilpotent ideal.*

Example 12. One might be worried about strictly upper triangular matrices, and strictly lower triangular matrices. So we can add these and take their span, but why is this not violating that the sum of nilpotent Lie ideals is a nilpotent Lie ideal? Neither of these are nilpotent ideals. They are somehow nilpotent, but not normal.

3.7. **Containments.** Note that all abelian Lie algebras are trivially nilpotent, but we also have that all nilpotent Lie algebras are solvable. Also note that trivially all simple Lie algebras are semisimple, and all semisimple Lie algebras are reductive. So being abelian and being simple are somehow two forms of "good" behavior that are just being generalized to get the other four types. In fact we have the following:

Lemma 1. *The intersection of semi-simple and solvable Lie algebras is empty.*

Proof. Let $\mathfrak{g}$ be semi-simple. Then it must be the direct sum of some simple Lie algebras $\mathfrak{g}_i$. It follows from linearity of the bracket, that

$$[\mathfrak{g}, \mathfrak{g}] = \bigoplus_i [\mathfrak{g}_i, \mathfrak{g}_i]$$

but since the $\mathfrak{g}_i$ are simple, they cannot have nonzero proper ideals, so the $[\mathfrak{g}_i, \mathfrak{g}_i]$ have to be trivial or the whole algebra, but if they were trivial then $\mathfrak{g}_i$ would be abelian, which is also not allowed. Therefore $[\mathfrak{g}_i, \mathfrak{g}_i] = \mathfrak{g}_i$ so $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ which prevents $[\mathfrak{g}, \mathfrak{g}]$ from being nilpotent, and therefore prevents $\mathfrak{g}$ from being solvable. $\square$

[3]This is also called the sol-radical, and is written $\mathcal{S}(\mathfrak{g})$.

Corollary 1. *The intersection of reductive and solvable Lie algebras consists of all abelian Lie algebras.*

Along a similar vein, we have the Levi decomposition, which says that the following sequence is split-exact:

$$0 \to \mathfrak{rad}\,(\mathfrak{g}) \to \mathfrak{g} \to \mathfrak{g}^{ss} \to 0$$

This sits in contrast with the following sequence which is always exact, but not necessarily split exact:

$$0 \to \mathfrak{nil}\,(\mathfrak{g}) \to \mathfrak{g} \to \mathfrak{g}^{\mathrm{red}} \to 0$$

where $\mathfrak{nil}\,(\mathfrak{g})$ is the nilradical of $\mathfrak{g}$ (the maximal nilpotent ideal) and $\mathfrak{g}^{\mathrm{red}}$ is some reductive Lie algebra.

Proposition 2. *If* $\mathfrak{nil}\mathfrak{g} \not\subseteq \mathfrak{rad}\mathfrak{g}$ *then* $\mathfrak{g}$ *is solvable.*

Proof. Take $\mathfrak{nil}\mathfrak{g} \oplus \mathfrak{rad}\mathfrak{g}$, this is solvable and strictly contains $\mathfrak{nil}\mathfrak{g}$ so it must be the whole thing and therefore must be solvable. ☐

4. Classification by dimension

We will classify one and two dimensional Lie algebras, and then we will focus on simple Lie algebras. In dimension 1, we have abelian $\mathbb{C}$, but every Lie algebra of dimension 1 is abelian.

In dimension 2, this can just be written $\mathfrak{g} = \mathbb{C}\langle x, y\rangle$. By definition, we know $[x,x] = [y,y] = 0$, and then we want to consider $[x,y] = -[y,x] = ax + by = z$.

$$[x, ax+by] = b\,(ax+by) \qquad\qquad [y, ax+by] = -a\,(ax+by)$$

which means $\mathbb{C}\langle ax+by\rangle \subseteq \mathfrak{g}$ is a Lie ideal, so either $a = b = 0$ or one of $a, b \neq 0$. In the first case we just have $\mathfrak{g} = \mathbb{C} \oplus \mathfrak{g}'$, but then $\mathfrak{g}'$ is of dimension 1, which means $\mathfrak{g}$ is abelian. Otherwise this is just some line, and WLOG we let $b \neq 0$. Now for $\mathfrak{g} = \mathbb{C}\langle x, z\rangle$ we get $[x,z] = bz$ so setting $x' = x/b$, we get $[x', z] = z$. In other words, any two-dimensional Lie algebra is either abelian, or has a basis $\{x', z\}$ such that $[x', z] = z$. As it turns out, this case is just:

$$\left\langle \begin{pmatrix} s & u \\ 0 & -s \end{pmatrix} \mid s, u \in \mathbb{C} \right\rangle$$

and in particular,

$$z = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad\qquad x' = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$$

In three dimensions we encounter our first simple Lie algebra, which is $\mathfrak{sl}\,(2, \mathbb{C})$.

LECTURE 9

LECTURE BY: PROFESSOR DAVID NADLER
NOTES BY: JACKSON VAN DYKE

1. Comments and corrections from last time

Recall semi-simple Lie algebras are direct sums of simple Lie algebras. Equivalently, $\mathfrak{rad}(\mathfrak{g})$, the sol-radical (maximal solvable ideal) is 0. Similarly $\mathfrak{g}$ is reductive iff $\mathfrak{rad}(\mathfrak{g}) = \mathfrak{z}$ is equal to its center.

We also said something about a Lie algebra being split by its center, which is not true in general. To see this, consider the following example:

Example 1. Let $\mathfrak{g} = \mathbb{C}\langle x, y, \kappa\rangle$ such that $[x, y] = \kappa$, $[x, \kappa] = [y, \kappa] = 0$. Clearly $\mathfrak{z} = \mathbb{C}\langle\kappa\rangle$, but there is no complement to the center which is closed under the bracket.

2. Representations of $\mathfrak{sl}(2, \mathbb{C})$

2.1. Motivation.

Recall we found that all dimension 1 Lie algebras are abelian, or just $\mathbb{C}$, and for dimension 2, we have either $\mathfrak{g} \cong \mathbb{C}^2$, or

$$\mathfrak{g} \cong \left\langle \begin{pmatrix} a & u \\ 0 & -a \end{pmatrix} \mid a, u \in \mathbb{C} \right\rangle$$

Now we move on to three dimensions.

We could play a similar game in dimension 3, but the interesting thing about 3 dimensions is that we get our first simple Lie algebra: $\mathfrak{sl}(2, \mathbb{C})$.

2.2. Preliminaries.

We will think of $\mathfrak{sl}(2, \mathbb{C})$ as having the following basis:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

where the brackets are:

$$[H, X] = 2X \qquad [H, Y] = 2Y \qquad [X, Y] = H$$

Exercise 1. Check these by hand.

Note $\mathfrak{sl}(2, \mathbb{C})$ by definition comes as 2×2 traceless matrices. Our generic goal here is to classify the matrix representations of Lie algebras such as $\mathfrak{sl}(2, \mathbb{C})$.

Recall:

Definition 1. A representation of a Lie algebra $\mathfrak{g}$ is a Lie algebra map $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ for a vector space $V/\mathbb{C}$. Recall $\mathfrak{gl}(V) = \mathrm{End}(V)$.

We will write the category of such representations as $\mathbf{Rep}(\mathfrak{g})$. This is an abelian category, which basically means we can do all of our friendly vector space operations to these things.

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Example 2. We always have the trivial one which is just $V = \mathbb{C}$ 1-dimensional and ρ the zero map.

Example 3. We also always have the adjoint representation where $V = \mathfrak{g}$, and $\rho = \mathrm{ad}$.

We will focus on the theory of finite dimensional representations, the category of which we write as $\mathbf{Rep}_{\mathrm{fd}}(\mathfrak{g})$. Note that this just means $\dim V$ is finite. This doesn't mean infinite dimensional ones aren't worth considering, but they have their own beautiful[1] story.

2.3. **Producing some representations.** We could start deductively, but we will instead start with some examples, and then see why we happen to end up with everything.

Example 4. Let $V_0 = \mathbb{C}$ be the trivial representation, so $\rho_0 = 0$. Let $V_1 = \mathbb{C}^2$, and ρ_1 be the inclusion $\mathfrak{sl}(2,\mathbb{C}) \hookrightarrow \mathfrak{gl}(2,\mathbb{C})$.

Now we will "generate" more representations using linear algebra. One thing we can always do, is take direct sums of representations. We will write $(V_1, \rho_1) \oplus (V_2, \rho_2) = (V_1 \oplus V_2, \rho_1 \oplus \rho_2)$ where $\rho_1 \oplus \rho_2$ acts via block matrices. This isn't really so interesting though.

We can also take the tensor product, which is very very interesting.[2]

$$(V_1, \rho_1) \otimes (V_2, \rho_2) = (V_1 \otimes V_2, \rho_1 \otimes \rho_2)$$

The definition of this map is as follows:

$$\rho_1 \otimes \rho_2 (x) = \rho_1 (x) \otimes \mathrm{id}_{V_2} + \mathrm{id}_{V_1} \otimes \rho_2 (x)$$

This definition effectively results from the Leibniz rule for differentiating the natural Lie group action on the tensor product. In particular, if we replace x with some $\gamma(t)$, we get

$$\rho_1 \otimes \rho_2 (\gamma(t)) (v_1 \otimes v_2) = \rho_1 (\gamma(t)) v_1 \otimes \rho_2 (\gamma(t)) v_2$$

and differentiating gives us the above definition.

Now let's calculate some tensor products.

Exercise 2. Tensoring with the trivial representation is the identity functor on **Rep**.

Let's tensor the standard representation (V_1, ρ_1) with itself. First let $V_1 = \mathbb{C}\langle e_1, e_2\rangle$ where e_i is the usual basis $(1,0)$ $(0,1)$. Then

$$V_1 \otimes V_1 = \mathbb{C}\langle e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2\rangle \ .$$

Now we calculate the action:

$$H (e_1 \otimes e_1) = (He_1) \otimes e_1 + e_1 \otimes (He_1) = e_1 \otimes e_1 + e_1 \otimes e_1 = 2 (e_1 \otimes e_1)$$

and similarly:

$$H (e_2 \otimes e_2) = -2 (e_2 \otimes e_2) \qquad H (e_1 \otimes e_2) = H (e_2 \otimes e_1) = 0$$

[1] And combinatorially complicated.

[2] Professor Nadler says that this if we remember only one thing, this should maybe be it.

Now since X annihilates e_1, we can calculate

$$X(e_1 \otimes e_1) = 0 \qquad X(e_1 \otimes e_2) = e_1 \otimes e_1$$
$$X(e_2 \otimes e_1) = e_1 \otimes e_1 \qquad X(e_2 \otimes e_2) = e_1 \otimes e_2 + e_2 \otimes e_1$$

and finally for Y, we have

$$Y(e_1 \otimes e_1) = e_2 \otimes e_1 + e_1 \otimes e_2 \qquad Y(e_1 \otimes e_2) = e_2 \otimes e_2$$
$$Y(e_2 \otimes e_1) = e_2 \otimes e_2 \qquad Y(e_2 \otimes e_2) = 0$$

If we order our basis as follows:

$$e_1 \otimes e_1 \qquad e_1 \otimes e_2 \qquad e_2 \otimes e_1 \qquad e_2 \otimes e_2$$

we can explicitly write:

$$H = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} \qquad X = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

Now we want to see if $V_1 \otimes V_1$ has any nontrivial proper subrepresentation. We can see that:

$$\mathbb{C}\langle e_1 \otimes e_1, e_2 \otimes e_2, e_1 \otimes e_2 + e_2 \otimes e_1 \rangle = \mathrm{Sym}^2(V_1)$$

is such a subrepresentation.

Remark 1. Recall that:

$$\mathrm{Sym}^2(V) := V \otimes V / (v \otimes w - w \otimes v)$$

for any vector space V. In general this is defined as:

$$\mathrm{Sym}^n(V) = V^{\otimes n} / (\cdots \otimes v_i \otimes v_{i+1} \otimes \cdots - \cdots \otimes v_{i+1} \otimes v_i \otimes \cdots)$$

Does $\mathrm{Sym}^2(V_1)$ have a complement? I.e. the following sequence is exact, but is it split?

$$0 \to \mathrm{Sym}^2(V_1) \to V_1 \otimes V_1 \to V_1 \otimes V_1 / \mathrm{Sym}^2(V_1) \to 0$$

Remark 2. Recall that (over $\mathbb{C}$) we always have the following splitting:

$$V^{\otimes 2} = \mathrm{Sym}^2(V) \oplus \wedge^2 V \tag{1}$$

which consists of the symmetric tensors, and the skew-symmetric tensors.

Exercise 3. Show that the splitting in (1) respects the bracket structure for any V.

Solution. Take an arbitrary $\mathfrak{g}$ representation (V, ρ) and consider the representation $(V^{\otimes 2}, \rho^{\otimes 2})$. First consider $v \otimes w \in \mathrm{Sym}^2 V$. For any $X \in \mathfrak{g}$, we have:

$$(\rho \otimes \rho)(X)(v \otimes w) = \rho(v) \otimes w + v \otimes \rho(w)$$
$$(\rho \otimes \rho)(X)(w \otimes v) = \rho(w) \otimes v + w \otimes \rho(v)$$

But since $v \otimes w = w \otimes v$, these are actually equal, so this is in $\mathrm{Sym}^2 V$ as well. Effectively the same argument holds for $\wedge^2 V$ by linearity.

So this does indeed have a complement, and this gives us another subalgebra

$$\wedge^2 (V) = \mathbb{C} \langle e_1 \otimes e_2 - e_2 \otimes e_1 \rangle$$

Now the question is, do we already know these by another name?

We know $\mathrm{Sym}^2 (V_1)$ is 3-dimensional, and $\wedge^2 (V_1)$ is 1-dimensional. As it turns out, we can check manually, that $\wedge^2 (V_1) = V_0$ is the trivial representation, but we also have the following:

Exercise 4. Any 1-dimensional representation of a semi-simple Lie algebra is trivial.

Solution. Consider the case of a simple Lie algebra. The kernel of ρ must contain the bracket, but if $\mathfrak{g}$ is simple, $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, so ρ must be trivial. But for semi-simple, we also have $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ so it holds for this case as well.

Now notice that $\mathrm{Sym}^2 (V_1)$ is just the adjoint representation. In fact, we can write down an isomorphism explicitly:

$$H \mapsto -(e_1 \otimes e_2 + e_2 \otimes e_1) \qquad X \mapsto e_1 \otimes e_1 \qquad Y \mapsto -e_2 \otimes e_2$$

Now we just have to check this map respects the action of the basis elements H, X, and Y. This map clearly respects the H action since the eigenvalues match. For the X action we can calculate:

$$\begin{aligned} \mathrm{ad}_X X &= 0 = X(-e_1 \otimes e_1) \\ \mathrm{ad}_X Y &= H \mapsto -(e_1 \otimes e_2 - e_2 \otimes e_1) = X(-e_2 \otimes e_2) \\ \mathrm{ad}_X H &= -2X \mapsto -2e_1 \otimes e_1 = X(-(e_1 \otimes e_2 + e_2 \otimes e_1)) \end{aligned}$$

Finally, we have to check the action of Y:

$$\begin{aligned} \mathrm{ad}_Y (Y) &= 0 = Y(-e_2 \otimes e_2) \\ \mathrm{ad}_Y (X) &= -\mathrm{ad}_X (Y) = -H \mapsto e_2 \otimes e_2 + e_1 \otimes e_2 = Y(e_1 \otimes e_1) \\ \mathrm{ad}_Y (H) &= -\mathrm{ad}_H (Y) = 2Y \mapsto -2e_2 \otimes e_2 = Y(-(e_1 \otimes e_2 + e_2 \otimes e_1)) \end{aligned}$$

So this does indeed preserve the action of the basis of $\mathfrak{sl}(2, \mathbb{C})$.

2.4. **General story.**

Definition 2. A semi-simple category is a category such that all objects are a direct sum of irreducible objects.

Here irreducible means there are no nontrivial proper subrepresentations.

Theorem 1. *The category of finite dimensional representations,* $\mathbf{Rep}_{fd}(\mathfrak{sl}(2, \mathbb{C}))$, *is a semi-simple category. The irreducible representations are all of the form* $V_n = \mathrm{Sym}^n (V_1)$ *for* $n \in \mathbb{N}$.

Note that V_0 is trivial, V_1 is standard, V_2 is adjoint, and the rest don't have names.

Lemma 1 (Schur). *Let* V_1 *and* V_2 *be irreducible representations of some Lie algebras* $\mathfrak{g}$*, then*

$$\mathrm{Hom}_{\mathbf{Rep}(\mathfrak{g})} (V_1, V_2) = \begin{cases} \langle 0 \rangle & V_1 \ncong V_2 \\ \mathbb{C} & V_1 \cong V_2 \end{cases}$$

Exercise 5. Prove this. This doesn't have much to do with Lie algebras and is more related to abelian categories.

Remark 3. Some aspects of this theorem generalize, for example $\mathbf{Rep}_{\mathrm{fd}}(\mathfrak{g})$ is a semisimple category iff $\mathfrak{g}$ is semisimple.

We now explain some structure we will use in the proof next time. Our strategy for understanding all representations, is to first hope and pray it is abelian, and if not we can just look at the diagonals and build up from there. Accordingly we first focus on a subalgebra $\mathfrak{h} = \mathbb{C}\langle H\rangle \subseteq \mathfrak{sl}(2,\mathbb{C})$. This is a 1-dimensional abelian Lie algebra, and

$$\mathbf{Rep}_{\mathrm{fd}}(\mathfrak{h}) = \mathbb{C}[H]\text{-}\mathbf{Mod}_{\mathrm{fd}}$$

so every such representation is just a choice of a vector space, and a choice of endomorphism $H \circlearrowright V$.

Recall the classification of such things uses Jordan forms, so block matrices with a generalized eigenvalue along the diagonal, and 1 along the super diagonal. We can picture this as a complex plane, where we have attached a generalized eigenspace at each λ_i:

$$V = \bigoplus_{\lambda_i} V_{\lambda_i}$$

Next time, we will take $\mathfrak{sl}(2,\mathbb{C})$ and see how the other operators interact with this picture.

LECTURE 10
MATH 261A

LECTURE BY: PROFESSOR DAVID NADLER
NOTES BY: JACKSON VAN DYKE

1. Representations of $\mathfrak{sl}(2,\mathbb{C})$

1.1. **Motivation.** We might wonder if we have a presentation of an algebra as matrices, why we care about additional representations? For example, we have a standard representation of $\mathfrak{sl}(2,\mathbb{C})$, so why do we care about anything else?

The point is, this isn't all about $\mathfrak{sl}(2,\mathbb{C})$. Of course we do understand $\mathfrak{sl}(2,\mathbb{C})$, but what we really want to understand is geometric representations coming from actions of $\mathfrak{sl}(2,\mathbb{C})$. So we really want to develop Lie groups as a tool rather than something to be studied.

Example 1. Let $X \subseteq \mathbb{CP}^n$ be a smooth projective variety over $\mathbb{C}$. One of the most important invariants we can associate to X is the cohomology $H^*(X,\mathbb{C})$, which is a vector space which has something to do with X. Then we have the following theorem:

Theorem 1 (Hard Lefschetz)**.** *$H^*(X,\mathbb{C})$ is naturally an $\mathfrak{sl}(2,\mathbb{C})$ representation.*

1.2. **Classification.** Recall we were about to prove the following last time:

Theorem 2. $\mathbf{Rep}_{fd}(\mathfrak{sl}(2,\mathbb{C}))$ *is semisimple, and the irreducibles are*

$$V_n = \mathrm{Sym}^n(V_1)$$

where V_1 is the standard representation.

Proof. Inside $\mathfrak{sl}(2,\mathbb{C}) = \mathfrak{g}$, consider the subalgebra $\mathfrak{h} = \mathbb{C}\langle H\rangle \subseteq \mathfrak{g}$. This is a one-dimensional abelian subalgebra.[1] Finite dimensional representations of $\mathfrak{h}$ are the same as finite dimensional vector spaces with an endomorphism:

$$\mathbf{Rep}_{\mathrm{fd}}(\mathfrak{h}) = \langle H \circlearrowleft V \mid H \in \mathrm{Ext}(V)\rangle \simeq \mathbb{C}[H]\text{-}\mathbf{Mod}$$

Every time you see a module over a polynomial algebra, you should think of the eigenline. So think of $\mathbb{C}$ as the eigenline of H. Then

$$V = \bigoplus V_{\lambda_i}$$

Now what can we say about representations of $\mathfrak{h}$ that come from $\mathfrak{g}$? This is not sort of mathematically canonical, but our strategy will be to consider the real direction as special. So project the eigenline to $\mathbb{R}$, which is of course ordered, which will allow us to analyze this picture from right to left.

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[1] In particular it is maximal, which makes it a Cartan subalgebra by definition. We will be seeing these later.

Definition 1. We will call the eigenvalues of an $\mathfrak{h}$ representation the *weights*. The *highest weight* will be the weight with real part $\geq$ the others. Call any vector in the eigenspace V_{λ_i} of the highest weight λ_i a *highest weight vector*. A general $v \in V_\lambda$ is said to be of weight λ.

Now bring X and Y into the picture. We now make a fundamental observation. Suppose $v \in V$ is of weight λ. Let's apply X and Y to v. The weight of $X \cdot v$ is just the eigenvalue of $X \cdot v$ under the action of H. But we can write:

$$HXv = XHv + [H, X]\, v = XHv + 2Xv$$

If $Hv = \lambda v$, i.e. v is an eigenvector, then

$$HXv = X\lambda v + 2Xv = (2 + \lambda)\, Xv$$

so it just shifted the eigenvalue by 2. Similarly, Y shifts the eigenvalue by -2.

Now we have to make sure nothing goes wrong when we act X and Y on the generalized eigenvectors.

Exercise 1. Show that if $(H - \lambda I)^n\, V = 0$, then

$$(H - (\lambda + 2)\, I)^n\, Xv = 0$$

and similarly for Y.

Solution. Proceed by induction. So suppose $(H - \lambda I)^{n-1}\, Xv = 0$. Then we can write:

$$\begin{aligned}
(H - \lambda I)^n\, Xv &= (H - \lambda I)^{n-1}\, (HXv - \lambda IXv) \\
&= (H - \lambda I)^{n-1}\, ((2 + \lambda)\, Xv - \lambda Xv) \\
&= 2\, (H - \lambda I)^{n-1}\, Xv = 0
\end{aligned}$$

as desired.

So in conclusion, $X : V_\lambda \to V_{\lambda+2}$ and $Y : V_\lambda \to V_{\lambda-2}$.

Now we want to use this to find the irreducibles. Suppose V is a finite dimensional irreducible $\mathfrak{sl}(2, \mathbb{C})$ representation. The first step is to find a highest weight λ_{hw}, and choose some eigenvector $v_{\text{hw}} \in V_{\lambda_{\text{hw}}}$.

Remark 1. This exists, because of the following. When you look at a Jordan block, the first vector is an eigenvector. So it doesn't matter if λ_{hw} yields an eigenspace or a generalized eigenspace, since there will still be an eigenvector either way.

If H was the only operator, this would be irreducible since $Hv_{\text{hw}} = \lambda_{\text{hw}} v_{\text{hw}}$. But now we have X and Y as well. Since λ_{hw} is the highest weight[2], $Xv_{\text{hw}} = 0$. Now start applying Yv_{hw} to get something in $V_{\lambda_{\text{hw}}-2}$, and continue applying Y. Of course since V is finite dimensional, this will eventually terminate.

Claim 1. The vectors

$$\mathbb{C}\left\langle v_{\text{hw}}, Yv_{\text{hw}}, Y^2 v_{\text{hw}}, \cdots \right\rangle \subseteq V$$

comprise an irreducible representation. In particular, if V is irreducible, then this is an equality.

[2] Professor Nadler says you should be yelling highest weight in your sleep.

Proof. The first thing this is saying is that these vectors span a subspace of the representation which is invariant under the operators. It is clear that H and Y preserve this so we need to show X preserves it. Of course $Xv_{\text{hw}} = 0$, and

$$XYv_{\text{hw}} = \cancel{YXv_{\text{hw}}} + [X, Y]\, v_{\text{hw}} = Hv_{\text{hw}} = \lambda_{\text{hw}} v_{\text{hw}}$$

which is in this subspace.

Exercise 2. Iterate this process.

Solution. Proceed by induction. So suppose $XY^{n-1}v_{\text{hw}} \in \left\langle Y^i v_{\text{hw}} \right\rangle$. Then we can write:

$$\begin{aligned} XY^n v_{\text{hw}} &= YXY^{n-1}v_{\text{hw}} + [X, Y]\, Y^{n-1}v_{\text{hw}} \\ &= YXY^{n-1}v_{\text{hw}} + HY^{n-1}v_{\text{hw}} \in \mathbb{C}\left\langle Y^i v_{\text{hw}} \right\rangle \end{aligned}$$

as desired.

It is clear that this is irreducible, because if you defined any sort of proper nontrivial subspace it would not be closed under the action of Y. $\square$

Next we will analyse the possible weight spaces. To do this, we will introduce universal highest weight modules.[3]

Definition 2. A *Verma* module I_λ of highest weight λ is

$$I_\lambda = \mathcal{U}\left(\mathfrak{sl}(2, \mathbb{C})\right) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_\lambda$$

Note that this is infinite dimensional.

Recall we had an adjunction where $\mathcal{U}$ was adjoint to Forget : **Lie-Alg** $\to$ **Alg**. This means

$$\operatorname{Hom}_{\mathbf{Alg}}\left(\mathcal{U}\mathfrak{g}, A\right) = \operatorname{Hom}_{\mathbf{Lie\text{-}Alg}}\left(\mathfrak{g}, \operatorname{Forget}(A)\right)$$

As a special case, for $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$, this means n-dimensional $\mathfrak{g}$ representations are just n-dimensional $\mathcal{U}\mathfrak{g}$-modules.

Recall that we can explicitly write the enveloping algebra as:

$$\mathcal{U}\mathfrak{g} = \bigoplus_{n=0}^{\infty} \mathfrak{g}^{\otimes n} / \left(XY - YX = [X, Y]\right)$$

The idea here is that $\mathcal{U}\mathfrak{g}$ allows us to work with products of operators of $\mathfrak{g}$.

The subalgebra $\mathfrak{b} \subseteq \mathfrak{sl}(2, \mathbb{C})$ is a Borel subalgebra:

$$\mathfrak{b} = \mathbb{C}\langle H, X\rangle = \left\langle \begin{pmatrix} a & u \\ 0 & -a \end{pmatrix} \mid a, u \in \mathbb{C} \subseteq \mathfrak{sl}(2, \mathbb{C}) \right\rangle$$

Note that this is a maximal solvable subalgebra.

Finally $\mathbb{C}_\lambda$ is the one dimensional complex vector space with one vector v such that H acts by multiplication by λ and X acts as 0:

$$Hv = \lambda v \qquad\qquad Xv = 0$$

In other words, the $\mathbb{C}_\lambda$ comprise the irreducible representations of $\mathfrak{b}$. In particular, these are all one-dimensional.

Exercise 3. Show that the $\mathbb{C}_\lambda$ comprise the irreducible representations of $\mathfrak{b}$.

[3] This might be against better judgement, but Professor Nadler says he just can't help himself.

Solution. First notice that $[\mathfrak{b}, \mathfrak{b}]$ must be inside $\ker \rho$. Since $[\mathfrak{b}, \mathfrak{b}] = \mathbb{C}\langle X \rangle$, this means X must act trivially. Then we know there must be some λ eigenvalue of H, so we can decompose this space if it is not of a single dimension.

The point here, is that we take the enveloping algebra, and then every time we see an H or an X, we can act by these rules and cancel.

Claim 2. I_λ has basis v, Yv, ...

Remark 2. This is a special case of the Poincaré-Birkhoff-Witt (PBW) theorem.

Proof. Look at some monomial. Using the bracket, we can rewrite this as a sum of monomials of the form $Y^a H^b X^c$.

So I_λ is spanned by vectors of the form $Y^a H^b X^c \otimes v$. For $c \neq 0$, we can use that we are tensoring over $\mathcal{U}(\mathfrak{b})$ to move X to the other side:

$$Y^a H^b X^c \otimes v = Y^a H^b X^{c-1} \otimes Xv = 0$$

So we may as well assume $c = 0$, and if $b \neq 0$,

$$Y^a H^b \otimes v = Y^a H^{b-1} \otimes Hv = \lambda \left(Y^a H^{b-1} \otimes v \right)$$

□

We do a sample calculation in V_λ to see the flavor of this:

$$\begin{aligned} XY^2 v &= XYYv = (YX + [X,Y]) Yv = YXYv + HYv \\ &= Y(YX + [X,Y])v + (YH + [H,Y])v \\ &= \cancel{Y^2 Xv} + \lambda Yv + \lambda Yv - 2Yv = \boxed{2(\lambda - 1) Yv} \end{aligned}$$

Lemma 1. *The action of H on the basis is given by:*

$$HY^j v = (\lambda - 2j) Y^j v$$

Proof. Proceed by induction. So assume

$$HY^{j-1} v = (\lambda - 2(j-1)) Y^{j-1} v$$

and then this allows us to write:

$$\begin{aligned} HY^j v &= (YH + [H,Y]) Y^{j-1} v \\ &= YHY^{j-1} v - 2Y^j v \\ &= Y\left(\left(\lambda - 2(j-1) Y^{j-1} \right) \right) - 2Y^j v \\ &= (\lambda - 2(j-1)) Y^j - 2Y^j \\ &= (\lambda - 2j) Y^j \end{aligned}$$

as desired. □

Lemma 2. *The action of X on the basis is given by:*

$$XY^j v = j(\lambda - (j-1)) Y^{j-1} v$$

Proof. Proceed by induction. So assume

$$XY^{j-1}v = (j-1)(\lambda - (j-2))\, Y^{j-2}v$$

then this lets us write:

$$\begin{aligned} XY^{j}v &= (YX + [X,Y])\, Y^{j-1}v \\ &= YXY^{j-1}v + HY^{j-1}v \end{aligned}$$

now we can rewrite each of these terms. First, using the induction hypothesis we can write:

$$\begin{aligned} YXY^{j-1}v &= Y(j-1)(\lambda - (j-2)) \\ &= ((j-1)\lambda - (j-2)(j-1))\, Y^{j-1}v \end{aligned}$$

and using lemma 1 we can write:

$$HY^{j-1}v = (\lambda - 2(j-1))\, Y^{j-1}v$$

which means

$$\begin{aligned} XY^{j}v &= ((j-1)\lambda - (j-2)(j-1) - \lambda + 2(j-1))\, Y^{j-1}v \\ &= j(\lambda - (j-1))\, Y^{j-1}v \end{aligned}$$

as desired. □

Remark 3. The basic idea of Verma modules is to somehow get a universally non-terminating object.

So for each λ there is this Verma module as defined above, but now this in fact has the universal property:

Exercise 4. Check that if λ is the highest weight:

$$\mathrm{Hom}_{\mathbf{Rep}(\mathfrak{g})}(I_\lambda, V) = \lambda \text{ eigenspace}$$

Solution. Let $f \in \mathrm{Hom}_{\mathbf{Rep}_{\mathrm{fd}}(\mathfrak{g})}(W_\lambda, V)$. This is completely determined by where it takes the basis of W_λ, and in particular, since this must respect the action of $\mathfrak{g}$, it is completely specified by where it takes the vector v. It must take this to some element of V_λ in order to preserve the action of H, and therefore we can associate f to the image $f(v) \in V_\lambda$.

This is a kind of standard adjunction, where we ask for the Hom of V_λ to any V, be the same as a Hom from $\mathbb{C}_\lambda$ as a $\mathcal{U}(\mathfrak{b})$ module. So we need to find vectors which are killed by X, and for which H acts as λ.

Now we return to representations of $\mathfrak{sl}(2,\mathbb{C})$. We have a canonical map $V_{\lambda_{\mathrm{hw}}} \to V$ which simply sends $v \to v_{\lambda_{\mathrm{hw}}}$. There must be some kernel, since this is a map from an infinite dimensional thing to a finite dimensional thing.

Claim 3. If we similarly generate from some different $v'_{\mathrm{hw}} \in V'_{\lambda'_{\mathrm{hw}}}$ in some representation V', we obtain isomorphic irreducible subspaces.

I.e. there is somehow no ambiguity. The isomorphism is the obvious one.

To be continued next time. . .

■

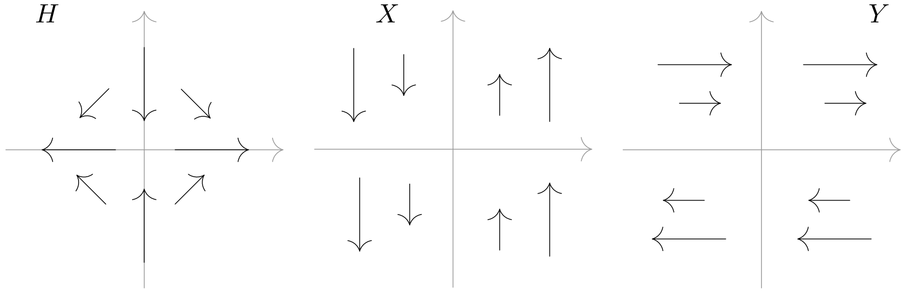


FIGURE 1. The vector fields which H, X, and Y are mapped to. The first can be thought of as being sort of hyperbolic, and the second two are shears.

1.3. Where does this come from. Recall

$$\mathrm{Sym}^n(W) \subset W^{\otimes n}$$

consists of the Σ_n-symmetric tensors.

How would someone come up with theorem 2? Imagine we start with $\mathrm{SL}(2,\mathbb{C}) \circlearrowright V_1 = \mathbb{C}^2 = \mathbb{C}\langle u, v\rangle$. Then we differentiate this to give us: $\mathfrak{g} \to \mathbf{Vect}\left(\mathbb{C}^2\right)$. In particular, calculate

$$\frac{d}{dt}\left(e^{tH}\cdot w\right)|_{t=0} = \frac{d}{dt}\begin{pmatrix} e^t & 0 \\ 0 & e^{-t}\end{pmatrix}\begin{pmatrix} w_1 \\ w_2\end{pmatrix}|_{t=0} = \begin{pmatrix} w_1 \\ -w_2\end{pmatrix}$$

so

$$H \mapsto u\partial_u - v\partial_v \qquad X \mapsto u\partial_v \qquad Y \mapsto v\partial_u$$

These vector fields can be visualized in fig. 1. Now consider the polynomial functions on $\mathbb{C}^2$, $\mathbb{C}[u,v]$, then these vector fields act on this, to get

$$\mathbb{C}[u,v] = \mathbb{C} \oplus \mathbb{C}\langle u, v\rangle \oplus \mathbb{C}\left\langle u^2, uv, v^2\right\rangle \oplus \cdots$$

which is effectively a decomposition in the irreducibles Sym^n.

LECTURE 11
MATH 261 A

LECTURE BY: PROFESSOR DAVID NADLER
NOTES BY: JACKSON VAN DYKE

1. Continued proof from last time

Recall we were in the middle of proving the following theorem:

Theorem 1. $\mathbf{Rep}_{fd}(\mathfrak{sl}(2,\mathbb{C}))$ *is semisimple, and the irreducibles are*

$$V_n = \operatorname{Sym}^n(V_1)$$

where V_1 is the standard representation.

Continued proof. Recall we're trying to use this weight picture to show this. Let's do some examples to get a feeling for this.

Example 1. The standard representation $\mathbb{C}\langle u, v\rangle$. This has eigenvalue -1 with eigenvector v, and 1 with eigenvector u.

Example 2. For $V_3 = \operatorname{Sym}^3(\mathbb{C}^2)$, we have the following eigen-vectors/values:

$$\lambda = -3, v_\lambda = v^3 \qquad \lambda = -1, v_\lambda = uv^2 \qquad \lambda = 1, v_\lambda = u^2 v \qquad \lambda = 3, v_\lambda = u^3$$

Recall the Verma module is:

$$I_\lambda = \mathcal{U}\mathfrak{sl}(2,\mathbb{C}) \otimes_{\mathcal{U}\mathfrak{b}} \mathbb{C}_\lambda$$

for $\mathfrak{b} = \mathbb{C}\langle H, X\rangle$. Also recall that we saw:

$$I_\lambda \simeq \mathbb{C}\left\langle v_\lambda, Y v_\lambda, Y^2 v_\lambda, \cdots\right\rangle$$

The Verma module also has the following universal property:

$$\operatorname{Hom}_{\mathfrak{g}}(I_\lambda, V) = \langle v \in V \mid Xv = 0, Hv = \lambda v\rangle$$

since v_λ has to go to something that is killed by X, and is an eigenvector of H with eigenvalue λ. The set on the RHS consists of highest weight vectors, and λ eigenvectors. In particular, if V is irreducible then there is a nonzero map $p : I_{\lambda_{\mathrm{hw}}} \to V$. This must be surjective because V is irreducible, and now we just need to figure out what the kernel is.

Proposition 1. *(1) If $\lambda \notin \{0, 1, 2, \cdots\} \subset \mathbb{C}$, then I_λ is irreducible.*
(2) For $n = 0, 1, \cdots$, there exists a short exact sequence:

$$0 \longrightarrow I_{-n-2} \hookrightarrow I_n \xrightarrow{p} V_n \longrightarrow 0$$

The first part implies that V irreducible must have $\lambda_{\mathrm{hw}} \in \{0, 1, 2, \cdots\}$. The second implies that once you get to $-n-2$, we see I_{-n-2} is sitting inside, so now we can quotient I_n / I_{-n-2} to get the finite dimensional representation V_n.

Date: October 2, 2018.

Warning 1. The sequence in the above proposition does not split.

Example 3. For $\lambda = 0$, we have: $I_{-2} \subset I_0$, now let's imagine if there's a complement of I_{-2} in I_0, but this can't be, since if we have v_0 and act by Y we immediately are moved out of this subspace.

Exercise 1. Prove the above proposition. The idea is to remember that $I_\lambda = \langle v_\lambda, Y v_\lambda, \cdots \rangle$, and then just apply X to see if there is any invariant subspace. So see if there's any way to come back.

Example 4. Let $\lambda = 0$. Then the basis of I_0 is $v_0, Y v_0, \cdots$ and $X v_0 = 0$, so we can calculate the following:

$$XYv_0 = \cancel{YXv_0} + [X, Y]\, v_0 = Hv_0 = 0v_0 = 0$$

However, as we saw last time:

$$XY^2 v_0 = (2\lambda - 2)\, Y v_0 = -2Y v_0$$

So once we have applied Y enough times, we reach the I_{-n-2} subspace, which in this case is I_{-2}. Then the quotient I_0/I_{-2} is the trivial representation.

Example 5. If $\lambda = 1$, we can calculate that:

$$XYv_1 = \cancel{YXv_1} + [X, Y]\, v_1 = Hv_1 = v_1$$

and similarly,

$$\begin{aligned} XY^2 v_1 &= (2\lambda - 2)\, Y v_1 = 0 \\ XY^3 v_1 &= 3\,(\lambda - 2)\, Y^2 v_1 = -3Y^2 v_1 \end{aligned}$$

So again, if we apply Y enough times we reach $I_{-n-2} = I_{-3}$, and then we can't get out. Then quotienting I_1/I_{-3} gives us the standard representation.

Exercise 2. Generalize these formulas.

Now we just need to prove the representations are semisimple. There are two approaches, one is kind of algebraic, and one is kind of geometric.[1] We will prove it the second way.

Recall we have an equivalence between simply connected, connected Lie groups over $\mathbb{C}$ and finite dimensional Lie algebras over $\mathbb{C}$. In particular, this means for any complex vector space V,

$$\mathrm{Aut}\,(V) = \mathrm{GL}\,(V) \mapsto \mathfrak{gl}\,(V) = \mathrm{End}\,(V)$$

This means, for our arbitrary $\mathfrak{g}$, we have

$$\begin{tikzcd} \mathrm{Hom}_{\textbf{Lie-Alg}}\,(\mathfrak{g}, \mathfrak{gl}\,(V)) \arrow[rr, "\simeq"] \arrow[dr] & & \mathrm{Hom}_{\textbf{Lie-Gp}}\,(G, \mathrm{GL}\,(V)) \arrow[dl] \\ & \textbf{Vect} & \end{tikzcd}$$

i.e.

$$\mathbf{Rep}_{\mathrm{fd}}\,(G) \cong \mathbf{Rep}_{\mathrm{fd}}\,(\mathfrak{g})$$

and this preserves the natural forgetful map to **Vect**. Actually to see this, we technically need the following:

[1] According to Professor Nadler, some sort of higher intelligence might prefer the algebraic approach, but he is not a higher intelligence, so we will take the sort of geometric approach. Also because we will see some interesting important math along the way.

Exercise 3. Show that since G is simply-connected, the map $\operatorname{Hom}(G, \mathrm{GL}(V)) \to \operatorname{Hom}\left(G, \widetilde{\mathrm{GL}}(V)\right)$ is the inverse of the projection in the following diagram:

$$\begin{tikzcd}
\operatorname{Hom}_{\textbf{Lie-Alg}}(\mathfrak{g}, \mathfrak{gl}(V)) \arrow[r, equal] \arrow[dr] & \operatorname{Hom}_{\textbf{Lie-Gp}}\left(G, \widetilde{\mathrm{GL}}(V)\right) \arrow[d, shift right] \\
 & \operatorname{Hom}_{\textbf{Lie-Gp}}(G, \mathrm{GL}(V)) \arrow[u, shift right, bend right]
\end{tikzcd}$$

so these things are all equal. I.e. show that if we have a homomorphism of a simply connected group, it naturally lifts to the universal cover. So we have the following diagram:

$$\begin{tikzcd}
 & \widetilde{\mathrm{GL}}(V) \arrow[d] \\
G \arrow[r] \arrow[ur, dashed] & \mathrm{GL}(V)
\end{tikzcd}$$

Remark 1. The previous exercise holds for any group, not just $\mathrm{GL}(V)$.

Now to finish the proof of the theorem, it is sufficient to prove the following:

Proposition 2. $\mathbf{Rep}_{fd}(\mathrm{SL}(2, \mathbb{C}))$ *is semisimple.*

We will first reduce this to an even easier statement.
Consider $\mathrm{SU}(2) \subseteq \mathrm{SL}(2, \mathbb{C})$. Recall $\mathrm{SU}(2)$ are the matrices

$$\left\langle \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \mid |\alpha|^2 + |\beta|^2 = 1 \right\rangle$$

which also preserve the standard hermitian inner product:

$$\langle a_1 e_1 + a_2 e_2, b_1 e_1 + b_2 e_2 \rangle = \overline{a}_1 b_1 + \overline{a}_2 b_2$$

Exercise 4. Show this is true.

Solution. Let $A \in \mathrm{SU}(2)$. Then

$$\langle Av, Aw \rangle = \overline{Av}^T Aw \overline{v}^T \overline{A}^T Aw = \overline{v} w = \langle v, w \rangle$$

where the last equality uses the fact that $\overline{A}^T A = I$.

Notice the following good properties of $\mathrm{SU}(2) \subset \mathrm{SL}(2, \mathbb{C})$:

(1) $\mathrm{SU}(2)$ is compact and isomorphic to S^3
(2) $\mathfrak{su}(2) \otimes \mathbb{C} \simeq \mathfrak{sl}(2, \mathbb{C})$

so the first says it is small, and the seccond says it's big in the sense that it doesn't miss any of the structure of $\mathfrak{sl}(2, \mathbb{C})$.

Remark 2. Subgroups of $G_{\mathbb{C}}$ with these properties are called "maximal compact". I.e. this doesn't really have anything to do with $\mathfrak{sl}(2, \mathbb{C})$.

Lemma 1. *The restriction*

$$\mathbf{Rep}_{fd}(\mathrm{SL}(2, \mathbb{C})) \xrightarrow{\sim} \mathbf{Rep}_{fd}(\mathrm{SU}(2))$$

is an isomorphism.

Proof. Since $\mathrm{SU}(2)$ is simply connected, $\mathbf{Rep}_{\mathrm{fd}}(\mathrm{SU}(2)) \simeq \mathbf{Rep}_{\mathrm{fd}}(\mathfrak{su}(2))$.

Exercise 5. Show the restriction:

$$\mathbf{Rep}_{\mathrm{fd}}(\mathfrak{su}(2) \otimes_{\mathbb{R}} \mathbb{C}) \xrightarrow{\sim} \mathbf{Rep}_{\mathrm{fd}}(\mathfrak{su}(2))$$

is an isomorphism.

This is effectively a tautology. So we are done. □

Proposition 3. $\mathbf{Rep}_{fd}(\mathrm{SU}(2))$ *is semisimple.*

Proof. Let V be a finite dimensional representation of $\mathrm{SU}(2)$. Then we will construct a hermitian inner product on V invariant under $\mathrm{SU}(2)$.

First choose any hermitian inner product $\langle v, w\rangle_0$. Now to make this invariant under the group action, we define:

$$\langle v, w\rangle = \int_{\mathrm{SU}(2)} \langle gv, gw\rangle_0 \, dg$$

Here, dg is a nonzero invariant measure[2] on $\mathrm{SU}(2)$. Suppose $W \subseteq V$ is a sub-representation, then we can consider

$$W^\perp := \{x \in V \mid \forall y \in W, \langle x, y\rangle = 0\}$$

Exercise 6. Show that $W^\perp \subseteq V$ is also a sub-representation, and in particular:

$$V \simeq W \oplus W^\perp$$

□

So the strategy was to go from a simple Lie algebra over $\mathbb{C}$, to a simply connected Lie group over $\mathbb{C}$, to maximal compact Lie group:

$$\mathfrak{g} \rightsquigarrow G \rightsquigarrow G_c$$

which all have the same representations. ■

1.1. **Invariant measure.** At the end of the proof of the above theorem we just asserted there was an invariant measure on $\mathrm{SU}(2)$. We now construct this. At every point of $\mathrm{SU}(2)$, we will define a volume form, i.e. a nondegenerate 3-form, and then this will give us a measure.

First pick an inner product on the tangent space at the identity. In particular, choose an Ad-invariant volume m on $\mathfrak{su}(2)$. One such example is the killing form. Now translate this by left multiplication to any $T_g\,\mathrm{SU}(2)$. Finally, observe that this is also right invariant. This is since the initial form was Ad invariant.

2. Playing with the representations

2.1. **Tensor products.** Take $V_n \otimes V_m$. This may or may not be irreducible, but it certainly will be a sum of irreducibles:

$$V_n \otimes V_m = \bigoplus_{k=0}^{\infty} V_k^{d_k} .$$

Then the challenge is to determined the d_k.

Example 6. $V_0 \otimes V_n = V_n$, so $d_n = 1$, and all other $d_i = 0$.

[2] See section 1.1.

Example 7. $V_1 \otimes V_1 = V_2 \oplus V_0$, so $d_0 = d_2 = 1$ and all other $d_i = 0$.

The past two examples were somehow easy to do without thinking too hard. The next example effectively generalizes to any case:

Example 8. Let's try to calculate $V_2 \otimes V_3$. The weights of V_2 are 2, 0, and -2. The weights of V_3 are 3, 1, -1, and -3. Then we observe that the restriction of a representation of Lie algebra to a Lie subalgebra, the tensor product is preserved. In particular, $\mathfrak{h} = \mathbb{C}\langle H \rangle \subseteq \mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ preserves $\otimes$.

Therefore the weights of $V_2 \otimes V_3$ are the pairwise sums of weights of V_2 and V_3 independently. Therefore the weights are:

$$-5 \qquad -3 \qquad -1 \qquad 1 \qquad 3 \qquad 5$$

where we have circled the weights as many times as their multiplicity. Then the multiplicity is how many ways these weights summed to give the new weights. Therefore the multiplicity of 5 is 1, the multiplicity of 3 is 2, the multiplicity of 1 is 3, and the same multiplicities for the negative weights.

Now we can understand the irreducibles just from this. Find the highest weight 5, then this means we must have a copy of V_5 inside, so we can cancel the weights associated with 5, so we have 1 left on ± 1, and 2 left on ± 1, so we have a V_3, and we cancel again, to get only 1 left on ± 1 so we get a V_1 and our answer is:

$$V_2 \otimes V_3 = V_5 \oplus V_3 \oplus V_1$$

Exercise 7. Write this down in general.

Solution. The basic idea is starting at the sum $m+n$ and then just counting down by 2 until you hit their difference. Let $m \geq n$, then:

$$V_m \otimes V_n = \bigoplus_{\substack{i \in 2\mathbb{Z} \\ m-n \leq i \leq m+n}} V_i$$

Next time we will generalize to all simple Lie algebras. In particular, we will write down a list of all such Lie algebras, and then see that the general story references $\mathfrak{sl}(2, \mathbb{C})$, so this is really an important thing to understand.

LECTURE 12
MATH 261A

LECTURES BY: PROFESSOR DAVID NADLER
NOTES BY: JACKSON VAN DYKE

We will meet 4 games people play with representations.

1. Tensor products

Recall last time we were playing some games with representations of $\mathfrak{sl}(2, \mathbb{C})$. In particular, we saw that for $m \geq n$,

$$V_m \otimes V_n = \bigoplus_{\substack{l=m-n+2k \\ 0 \leq k \leq n}} V_l$$

2. Characters

Consider $\mathbb{C}[\mathbb{Z}]$, the collection of compactly supported $\mathbb{C}$-valued functions[1] on $\mathbb{Z}$.

Definition 1 (Character)**.** A *formal character* is an element of $\mathbb{C}[\mathbb{Z}]$. We write e_n for the characteristic function of $n \in \mathbb{Z}$.

The e_n form a basis for $\mathbb{C}[\mathbb{Z}]$ as a complex vector space. This can be considered a ring with the operation given by convolution. This effectively just depends on the group structure on $\mathbb{Z}$.

$$(f * g)(n) = \sum_{k+l=n} f(k) g(l)$$

Exercise 1. Check that $e_n * e_m = e_{n+m}$.

This is somehow a linear extension of the group structure on $\mathbb{Z}$.

2.1. **More invariant origin.** Return to representation theory. We want to think about $\mathbb{Z}$ as integer weights in the H eigenline $\mathbb{C}$.

Definition 2 (Character of a representation)**.** The *formal character* of a finite dimensional representation V is $V \mapsto \chi_V \in \mathbb{C}[\mathbb{Z}]$ where

$$\chi_V(n) = \dim_{\mathbb{C}} V_{\lambda=n}$$

where $V_{\lambda=n}$ is the eigenspace at $\lambda = n$.

Example 1. The irreducibles from before have characters:

$$\chi_{V_n} = \sum_{\substack{l=-n+2k \\ 0 \leq k \leq n}} e_l$$

Date: October 4, 2018.

[1]Or maybe distributions. Or maybe not compactly supported. The mathematics is smart and will tell us what's right.

Exercise 2. Check the following:

$$\chi_{V \oplus W} = \chi_V + \chi_W \qquad \chi_{V \otimes W} = \chi_V * \chi_W$$

So characters somehow take a representation and return an element of $\mathbb{C}[\mathbb{Z}]$. But this statement isn't formal, since this is somehow mixing levels - representations are objects in a category, and $\mathbb{C}[\mathbb{Z}]$ is just a ring. One way to formalize this is to instead consider this as a map from the Grothendieck group to the ring of formal characters:

$$K_0\left(\mathbf{Rep}_{\mathrm{fd}}\left(\mathfrak{sl}(2,\mathbb{C})\right)\right) \otimes \mathbb{C} \to \mathbb{C}[\mathbb{Z}]$$

Recall **Rep** is an abelian category. The Grothendieck group is what you get when you ask for a group whose elements are the objects of your category, and direct sum becomes addition. It's somehow the universal version of a group resulting from only insisting that exact sequences make sense.

We can think of elements of this as being some sort of formal difference $V - W$ of two objects of the original category.

Proposition 1. *χ is injective, and in particular,*

$$\chi : K_0\left(\mathbf{Rep}_{fd}\mathfrak{sl}(2,\mathbb{C})\right) \otimes \mathbb{C} \xrightarrow{\sim} \mathbb{C}[\mathbb{Z}]^{\Sigma_2}$$

is an isomorphism, where $\Sigma_2 \simeq \mathbb{Z}/2$ acts by $\sigma(n) = -n$.

Proof. Injectivity follows from the fact that up to isomorphism, representations are determined by their characters. To see this is surjective, we just have to check that $e_n + e_{-n}$ is in the image, which is

$$\chi(V_n - V_{n-2})$$

so we are done. □

3. Character formulas

The game is the following. Put $n \in \mathbb{N}$ into the machine, and the machine is supposed to give you χ_{V_n}

$$\mathbb{N} \ni n \rightsquigarrow \chi_{V_n} \in \mathbb{C}[\mathbb{Z}]^{\Sigma_2}$$

The answer for $\mathfrak{sl}(2,\mathbb{C})$ is just the sum of the weights as above in example 1, but in general it won't be this easy. So we will consider in a complicated but beautiful way to do it for $\mathfrak{sl}(2,\mathbb{C})$ which will turn out to generalize.

We know we can take V_n which has a surjective map $I_n \to V_n$ from the Verma module, and in particular, we have the exact sequence:

$$0 \to I_{-n-2} \to I_n \to V_n \to 0$$

Remark 1. This is a special case of the Bernstein-Gelfand-Gelfand (BGG) resolution.

This sequence implies that the character[2] of V_n is the character of I_n minus the character of I_{-n-2}:

$$\chi_{V_n} = \chi_{I_n} - \chi_{I_{-n-2}}$$

[2] This isn't really a character because the dimension of each eigenspace to the infinite negative side has dimension 1 for these Verma modules. Therefore the associated character is not compactly supported.

But taking inspiration from:

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots$$

we can write this is a more clever way:

$$\chi_{V_n} = \frac{e_n}{1-e_{-2}} - \frac{e_{-n-2}}{1-e_{-2}} = \frac{e_n - e_{-n-2}}{1-e_{-2}}$$

Now rewriting this, we get:

$$\chi_{V_n} = \frac{e_{n+1} - e_{-n-1}}{e_1 - e_{-1}}$$

4. Tannakian formalism

Suppose $\mathbf{C}$ is a $\mathbb{C}$-linear abelian $\otimes$-category. Suppose

$$F : \mathbf{C} \to \mathbf{Vect}$$

is a "forgetful functor." This means this is a $\otimes$-functor which is exact, faithful, and maybe a few more things that the actual forgetful functor is.

To this, we can associate a group

$$G = G_{\mathbf{C},F} = \mathrm{Aut}^{\otimes}(F)$$

which is the group of tensor automorphisms of F. For $g \in G$, we get an automorphism

$$g_V : F(V) \xrightarrow{\sim} F(V)$$

for every $V \in \mathbf{C}$, which respects the tensor structure in the sense that:

$$g_{V \otimes W} = g_V \otimes g_W$$

This is called the Tannakian group of $\mathbf{C}$ with respect to the fiber functor F.

Exercise 3. For $\mathbf{C} = \mathbf{Rep}_{\mathrm{fd}}(\mathfrak{sl}(2,\mathbb{C}))$, and $F = \mathrm{Forget}$, then this says for every representation of $\mathfrak{sl}(2,\mathbb{C})$, forget it down to a vector space, then everything in $\mathrm{Aut}^{\otimes}(F)$ is a choice of automorphisms of these vector spaces. Show that $G_{\mathbf{C},F} \simeq \mathrm{SL}(2,\mathbb{C})$.

Solution. First start with an element $A \in \mathfrak{sl}(2,\mathbb{C})$, then we want to get an element $g \in G$, i.e. a collection of automorphisms

$$g_V \circlearrowright V \in \mathbf{Rep}_{\mathrm{fd}}(\mathfrak{sl}(2,\mathbb{C}))$$

It is enough to specify this on the irreducibles $V_n = \mathrm{Sym}^n V_1 = \mathrm{Sym}^n \mathbb{C}^2$.

Remark 2. If you have an abelian category you're trying to learn something about, try calculating the Tannakian group. By the above discussion, the category will then be the representations of this group, though the new group might be something terrible you've never seen before.

5. Classification of simple Lie algebras over $\mathbb{C}$

This is somehow the general answer over algebraically closed fields, but we will just do it over $\mathbb{C}$. This is called the Cartan classification.

5.1. **Classical Lie algebras.** The first type is A_n for $n \geq 1$, and these are

$$\mathfrak{sl}(n+1,\mathbb{C}) = \langle \operatorname{tr} A = 0 \rangle$$

The next series is B_n for $n \geq 2$, and these are the odd orthogonal Lie algebras

$$\mathfrak{so}(2n+1,\mathbb{C}) = \left\langle -A = A^T \right\rangle$$

This one starts at $\mathfrak{so}(5)$ because $\mathfrak{so}(3)$ is already on the list since:

Proposition 2. $\mathfrak{sl}(2) \simeq \mathfrak{so}(3)$

The next is C_n for $n \geq 3$, which are $\mathfrak{sp}(2n,\mathbb{C})$. These preserve the standard symplectic inner product:

$$\mathfrak{sp}(2n,\mathbb{C}) = \left\langle \omega A = -A^T \omega \right\rangle$$

for ω some nondegenerate skew-symmetric matrix/inner product. So these are linear automorphisms of a symplectic vector space.

Remark 3. We got this condition on elements of $\mathfrak{sp}$ by differentiating

$$(gv_1)^T \omega g v_2 = v_1^T \omega v_2$$

with respect to g which gives:

$$0 = (Av_1)^T \omega v_2 + v_1^T \omega A v_2 = A^T \omega + \omega A$$

For $n = 1$ we get $\mathfrak{sp}(2)$, which just consists of area preserving matrices, but this is $\mathfrak{sl}(2)$ so this is already on the list. And for $n = 2$ we have:

Exercise 4. Show that $\mathfrak{sp}(4) \simeq \mathfrak{so}(5)$.

Solution. *Proof.* Take (V,ω) to be a four-dimensional symplectic vector space. Then we have an action of $\mathrm{Sp}(4)$ on $\wedge^2 V$, which is 6-dimensional and preserves the symmetric pairing

$$\wedge^2 V \times \wedge^2 V \to \wedge^4 V = \mathbb{C}$$

So we have a map

$$\mathrm{Sp}(V) \to \mathrm{SO}\left(\wedge^2 V\right)$$

The element ω is fixed and its norm $\omega \wedge \omega \neq 0$, so $\mathrm{Sp}(V)$ fixes the 5-dimensional orthogonal complement $I^\perp$ and we have an induced map $\mathrm{Sp}(4,\mathbb{C}) \to \mathrm{SO}(5,\mathbb{C})$. Check it is surjective and at the level of Lie algebras induces the required isomorphism. □

Next we have D_n for $n \geq 4$ which corresponds to $\mathfrak{so}(2n,\mathbb{C})$. This indexing starts here because:

Proposition 3.

$$\mathfrak{so}(4,\mathbb{C}) \simeq \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}) \qquad \mathfrak{so}(6,\mathbb{C}) \simeq \mathfrak{sl}(4,\mathbb{C})$$

and

Proposition 4. $\mathfrak{so}(2,\mathbb{C})$ *is one-dimensional and commutative, and therefore it is not semisimple.*

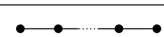
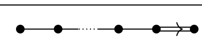
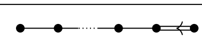
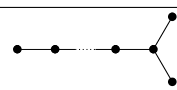
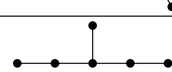
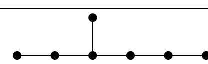
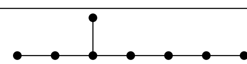
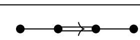
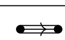
Note that the B_n and D_n are both $\mathfrak{so}(n)$ for odd/even n. So we could hypothetically lump these in the same list, but as we have seen, these algebras have different behaviours. In particular, the simply connected Lie groups corresponding to these Lie algebras have different centers. One has $\mathbb{Z}/4$, and one has $\mathbb{Z}/2 \times \mathbb{Z}/2$.

This is the full list of classical Lie algebras. Looking at this we can sort of ask what kinds of geometries we can do. And this tells us we can do classical euclidean geometry, which has to do with the orthogonal matrices, or you can do symplectic geometry, which of course has to do with the symplectic matrices.

5.2. **Exceptional Lie algebras.** Now we have the exceptional E_6, E_7, E_8, F_4, and finally G_2, and now this is everything.

Remark 4. In a certain sense, if you're a usual algebraist that likes to understand simple things and view them as atoms, these are somehow the atoms that things will be built out of.

5.3. **Dynkin diagrams.** We will come back to these later, but for now we will just see them as "hieroglyphics" which will help us remember this classification.

	$\mathfrak{g}$	Diagram	$Z(G)$	$\pi_1(G)$
$A_n\ (n \geq 1)$	$\mathfrak{sl}(n+1, \mathbb{C})$		$\mathbb{Z}/(n+1)\mathbb{Z}$	0
$B_n\ (n \geq 2)$	$\mathfrak{so}(2n+1, \mathbb{C})$		0	$\mathbb{Z}/2\mathbb{Z}$
$C_n\ (n \geq 3)$	$\mathfrak{sp}(2n, \mathbb{C})$		$\mathbb{Z}/2\mathbb{Z}$	0
$D_n\ (n \geq 4)$	$\mathfrak{so}(2n, \mathbb{C})$		$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
E_6	–		$\mathbb{Z}/3\mathbb{Z}$	–
E_7	–		$\mathbb{Z}/2\mathbb{Z}$	–
E_8	–		0	–
F_4	–		0	–
G_2	–		0	–

These appear all over mathematics.

Remark 5. One of the last things Grothendieck did before "leaving" mathematics to become a farmer, is that he found these Dynkin diagrams in resolutions of surface singularities. One can look at algebraic surfaces, and there are these nice classical du Val singularities, and they have natural resolutions, and then these diagrams show up in the geometry of their resolutions.[3]

These pictures bring to light a clear duality called Langlands duality, that isn't made apparent from the list itself. If we reverse the direction of the bar, then A_n, D_n, and E_n are self dual. The diagrams A_n, D_n, and E_n are called *simply-laced*. These are somehow the most basic ones. Then B_n and C_n are dual to one another. Then F_4 and G_2 are said to be *twisted self-dual*.

[3] We won't talk about this, but one can ask Professor Nadler some other time if one is interested.

One might want to play a game where we start with A_n, D_n, and E_n and recover B_n, C_n, F_4, and G_2 form some operations. There's a whole "game" called folding Lie algebras which allows you to take D_n, and sort of collapse the end together to get these double bars in B_n and C_n. Similarly, we can take D_5 and sort of collapse it down to F_4, and collapse D_4 into G_2.

5.4. **Associated groups.** In the table above we have written the centers of the associated Lie groups. Note however that these centers are of the "usual" group associated with the algebra. This is however not the unique simply connected one in the case of $\mathfrak{so}(2n+1)$ and $\mathfrak{so}(2n)$. In this case the unique simply connected one is $\mathrm{Spin}(2n+1)$ and $\mathrm{Spin}(2n)$ respectively. We list the centers so we can determine all of the groups which can be associated to these algebras since we just have to quotient out by subgroups of the center to get these.

In the case of A_n we can quotient out by any subgroup of $\mathbb{Z}/(n+1)\mathbb{Z}$, which is of course just any divisor of $n+1$. In the case of B_n we can take the universal cover, and then these are the only two: $\mathrm{SO}(2n+1)$ and $\mathrm{Spin}(2n+1)$. For C_n, we just get $\mathrm{Sp}(2n,\mathbb{C})$ and $\mathrm{Sp}(2n,\mathbb{C})/\mathbb{Z}/2$. Finally, for D_n we get $\mathrm{SO}(2n,\mathbb{C})$ and $\mathrm{SO}(2n,\mathbb{C})/\mathbb{Z}/2$, and $\mathrm{Spin}(2n)$. Only now $\mathrm{Spin}(2n)$ has center $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ if n is even and center $\mathbb{Z}/4\mathbb{Z}$ if n is odd.

Remark 6. One might wonder what Lie groups give rise to the exceptional Lie algebras. We can play the usual game, and take the adjoint representation, then since the algebras are simple, they have no center, so the adjoint representation puts it inside endomorphisms of some vector space, then we can exponentiate these matrices and get a group.

G_2 is the smallest, so it's sort of easiest to get our hands on. If we look at the unit octonions, we can then consider the automorphisms of the non-associative algebra of unit octonions, and this is G_2. In fact all of them arise as automorphisms of something. E_8 is probably the most important one in all of Math, it's somehow the biggest.

6. Finite dimensional representations of $\mathfrak{sl}(3,\mathbb{C})$

It's somehow the case that once one understands $\mathfrak{sl}(2,\mathbb{C})$, and then how to generalize this to $\mathfrak{sl}(3,\mathbb{C})$, there isn't much left to do to understand simple Lie algebras.

We want a similarly natural set of operators to act as a basis like we had for $\mathfrak{sl}(2,\mathbb{C})$. First we define:

$$H_{12}=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix}\qquad H_{23}\begin{pmatrix}0&0&0\\0&1&0\\0&0&-1\end{pmatrix}$$

These will again generate a subalgebra:

$$\mathfrak{h}=\mathbb{C}\langle H_{12},H_{23}\rangle\subseteq\mathfrak{sl}(3,\mathbb{C})$$

which is a 2-dimensional abelian subalgebra. Now we can consider all of the following matrices:

$$X_{12}\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}\qquad X_{13}=\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}\qquad X_{23}=\begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix}$$

$$Y_{21} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad Y_{31} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad Y_{32} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

We choose these since it's a basis of eigenvectors for $\mathfrak{h}$ acting on $\mathfrak{g}$ with the adjoint action.

LECTURE 13
MATH 261A

LECTURES BY: PROFESSOR DAVID NADLER
NOTES BY: JACKSON VAN DYKE

The second midterm will be Tuesday October 30th.

1. Root systems

1.1. **Cartan subalgebras.** The "biggest" abelian thing inside $\mathfrak{sl}(3, \mathbb{C})$ is generated by:

$$H_{12} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad H_{23} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

In particular, set $\mathfrak{h} = \langle H_{12}, H_{23} \rangle$.

Fact 1. *$\mathfrak{h}$ is a maximal abelian subalgebra. It also has the property that it is diagonalizable under the adjoint action* ad*.*

The fact that this is abelian means we can simultaneously diagonalize them. Such subalgebras are called *Cartan subalgebras*.

Warning 1. Though this is a convenient Cartan subalgebra it is not unique. However, as we will eventually see, this is actually unique up to conjugation.

1.2. **Roots.** We want to generalize the notion of an eigenvector/eigenvalue for one operator to an algebra. Write $\mathfrak{h}^*$ for the dual of $\mathfrak{h}$. This is the space of possible eigenvalues of $\mathfrak{h}$. Explicitly:

$$\mathfrak{h}^* = \{\lambda : \mathfrak{h} \to \mathbb{C} \text{ linear}\}$$

i.e. in higher dimensions, we should think of eigenvalues as being elements of the dual space.

Define L_1 to be a complex valued function on $\mathfrak{h}$ as follows:

$$L_1(H) = (1,1) \text{ entry of } H$$

for example $L_1(H_{12}) = 1$, and $L_1(H_{23}) = 0$. Define L_2 and L_3 similarly. Note that $L_1 + L_2 + L_3 = 0$.

If we consider $\mathfrak{h} \subseteq \mathbb{C}^3$, then

$$\mathfrak{h}^* = \left(\mathbb{C}^3\right)^* / \mathbb{C} \langle (1,1,1) \rangle$$

where we have quotiented out by the diagonal. We can sort of think of this like looking at the corner of a room as in fig. 1. We will use L_1, L_2, and L_3 as a basis of the dual space.

Date: October 9, 2018.

Now we restrict the adjoint representation to $\mathfrak{h}$. For $H \in \mathfrak{h}$, consider the operator $\mathrm{ad}_H : \mathfrak{g} \to \mathfrak{g}$. First let's fix a basis of eigenvectors.

$$X_{12} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad X_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad X_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$Y_{21} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad Y_{31} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad Y_{32} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Definition 1. The nonzero eigenvalues of $\mathrm{ad}|_{\mathfrak{h}} \circlearrowright \mathfrak{g}$ are called *roots*.

Exercise 1. Check these are eigenvectors.

Solution. We check X_{12} first. Since we are taking H_{12}, H_{23} as our basis for $\mathfrak{h}$, and since $\mathrm{ad}_H = [H, -]$, we need to calculate:

$$[H_{12}, X_{12}] = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 2X_{12}$$

$$[H_{23}, X_{12}] = 0 - \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -X_{12}$$

so we need to find an element of $\mathfrak{h}^*$ which maps $H_{12} \mapsto 2$, and $H_{23} \mapsto -1$. In particular, $L_1 - L_2$ is the root. We write this as α_{12}. The picture here is as in fig. 1.

For X_{13} we have:

$$[H_{12}, X_{13}] = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - 0 = X_{13}$$

$$[H_{23}, X_{13}] = 0 - \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -X_{13}$$

so this has root $L_1 - L_3$. A similar calculation holds for the remaining X_{ij} and Y_{ij}.

So from either brute force or cleverness we get that the roots are all $\alpha_{ij} = L_i - L_j$ for $i \neq j$. These form a hexagon as in fig. 1.

1.3. **Fundamental calculation.**

The following lemma plays the role of the "fundamental calculation" that we saw in the $\mathfrak{sl}(2, \mathbb{C})$ case.

Lemma 1. *Suppose V is a representation of $\mathfrak{sl}(3, \mathbb{C})$, and $v \in V$ is an $\mathfrak{h}$ eigenvector with eigenvalue $\lambda \in \mathfrak{h}^*$. Then $X_{ij} v$ is again an $\mathfrak{h}$ eigenvector with eigenvalue $\lambda + \alpha_{ij}$ for $i < j$. Similarly, $Y_{ij} v$ is again an eigenvector with eigenvalue $\lambda + \alpha_{ij} = \lambda - \alpha_{ji}$. for $i > j$*

According to this lemma, the X_{ij}s and Y_{ij}s have sort of "preferred" directions. There is a sort of X-cone which sweeps clockwise between α_{23} and α_{12}, and there is a Y-cone which sweeps clockwise between α_{32} and α_{21}.

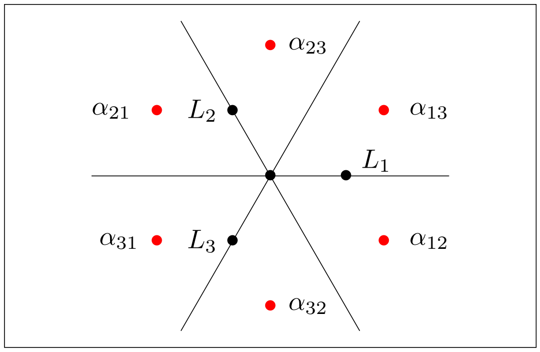


FIGURE 1. The real projection of $\mathfrak{h}^*$. The roots (in red) form a hexagon.

1.4. **Borel subalgebra and positive roots.** Now consider the subalgebra:

$$\mathfrak{b} := \mathfrak{h} + \mathbb{C}\langle X_{ij} \rangle$$

Note $\mathfrak{b}$ is a maximal solvable subalgebra. This makes sense since we somehow know solvable algebras to be upper triangular, and this is upper triangular. This is an example of a *Borel subalgebra.*[1] We will call the roots inside $\mathfrak{b}$ the *positive roots.* These are α_{23}, α_{13}, and α_{12}. We will write the collection of these as R^+.

1.5. **Simple roots.** Notice that $\alpha_{13} = \alpha_{23} + \alpha_{12}$. This somehow indicates that the roots α_{23} and α_{12} are more special. We will call these roots the simple roots. We will write the collection of simple roots as Δ^+. We will eventually see the following fact:

Fact 2. *All of the roots can be recovered from the simple roots.*

2. Representations of $\mathfrak{sl}(3, \mathbb{C})$

Now we're finally ready to meet some representations. Recall in the $\mathfrak{sl}(2)$ case the irreducibles were indexed by the natural numbers. We now meet the analogous object.

Definition 2. The dominant (integral) weights are:

$$\Lambda^+ = \mathbb{Z}_{\geq 0}\langle L_1, -L_3 \rangle$$

This is an integer lattice of L_1 and $-L_3$ as in fig. 2.

Then the theorem is as follows:

[1] Professor Nadler says that often times in mathematics, when one does something important, ones name becomes a noun forever, however when alive, people typically don't call these objects their own name. For example Hitchin himself never referred to a Hitchin system as such. A more relevant example is that Borel always just called this a "maximal solvable subalgebra" rather than use his own name. Since a choice of such a subalgebra is often accompanied by the choice of a Cartan subgroup, which has something to do with a torus, it is sometimes said that choosing such an $\mathfrak{h}$ and $\mathfrak{b}$ is a choice of a "borus". Professor Nadler wonders if Borel would have preferred this...

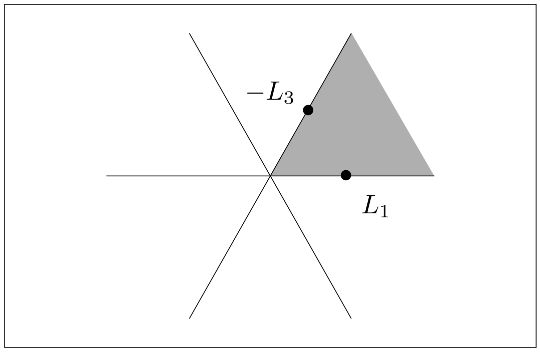


FIGURE 2. Dominant weights for $\mathfrak{sl}(3,\mathbb{C})$.

Theorem 1. *The finite dimensional representations of $\mathfrak{sl}(3,\mathbb{C})$ form a semisimple category $\mathbf{Rep}_{fd}(\mathfrak{sl}(3,\mathbb{C}))$, and the irreducibles are indexed by Λ^+:*

$$\Lambda^+ \ni \lambda \rightsquigarrow V_\lambda \in \mathbf{Rep}_{fd}(\mathfrak{sl}(3,\mathbb{C}))$$

where V_λ is some irreducible representation. We can reverse this construction by taking the highest weights with respect to $\mathfrak{b}$.

3. Constructing irreducible representations

Example 1. First we have $V_0 = \mathbb{C}$ is the trivial representation. The picture is just a single weight at 0.

Example 2. $V_{L_1} = \mathbb{C}^3$ will be the standard representation.

$$H = \begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix} \circlearrowright \mathbb{C}^3$$

for $a+b+c=0$. The eigenvectors are e_1, e_2, and e_3 which go to ae_1, be_2, and ce_3. The eigenvalues are $L_1(H)=a$, $L_2(H)=b$, and $L_3(H)=c$. The weights are just the L_1, L_2, L_3 that we have seen. Since $X_{ij}e_1 = 0$ for $i<j$ we see that L_1 is the highest weight.

Example 3. The representation $V_{-L_3} = \mathbb{C}^3$ is dual to the standard representation. The weights are as in fig. 3.

Example 4. Now consider the representation $V_{L_1-L_3} = V_{\alpha_{13}}$. We might guess that this is the tensor product $V_{L_1} \otimes V_{-L_3}$. Just like for $\mathfrak{sl}(2,\mathbb{R})$, the eigenvectors of the tensor product are tensors of the eigenvectors, so the weights just add as in fig. 4. Note that this is a nine-dimensional representation. Since V_{-L_3} is the dual of V_{L_1}, we have:

$$V_{L_1} \otimes V_{-L_3} = V_{L_1} \otimes V_{L_1}^* = \mathrm{Hom}(V_{L_1}, V_{L_1})$$

which in particular contains an invariant subspace $\mathbb{C}\langle \mathrm{id}_{V_{L_1}} \rangle$. From this point of view the identity is:

$$\mathrm{id}_{V_{L_1}} = e_1 \otimes e_1^* + e_2 \otimes e_2^* + e_3 \otimes e_3^*$$

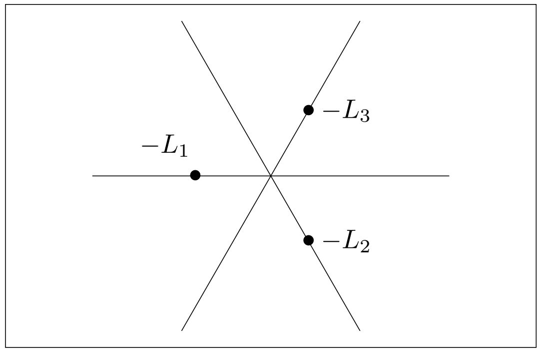


FIGURE 3. The roots of V_{-L_3}.

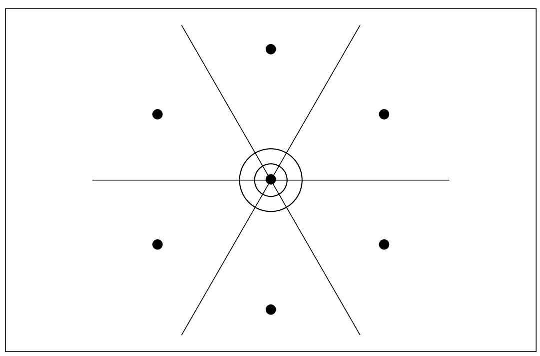


FIGURE 4. Roots of $V_{L_1} \otimes V_{-L_3}$.

This is like the diagonal matrices. Now we can decompose this the tensor into the invariant portion and whatever is left, which we are guaranteed to be a subspace since this is semisimple. The invariant portion just has one weight at 0, and then we are left with an eight-dimensional representation with the same weights, only one less multiplicity at 0. But we can recognize this eight-dimensional representation as the adjoint one, which means $V_{L_1-L_3} = \mathfrak{sl}(3,\mathbb{C})$ is the adjoint representation.

Now the story continues as it did in the case of $\mathfrak{sl}(2,\mathbb{C})$. We can keep tensoring the standard and adjoint representations until we get one that has a desired highest weight, and then we just have to decompose it to find the desired representation.

Example 5. Consider V_{2L_1}. Our first guess might be $V_{L_1} \otimes V_{L_1}$. The weights for this representation will be as in fig. 5. This is not irreducible, since as usual we can write:

$$V_{L_1} \otimes V_{L_1} = \mathrm{Sym}^2 (V_{L_1}) \oplus \wedge^2 (V_{L_1})$$

Then Sym^2 has the weights

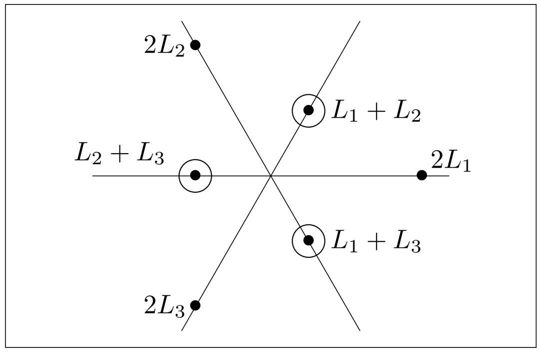
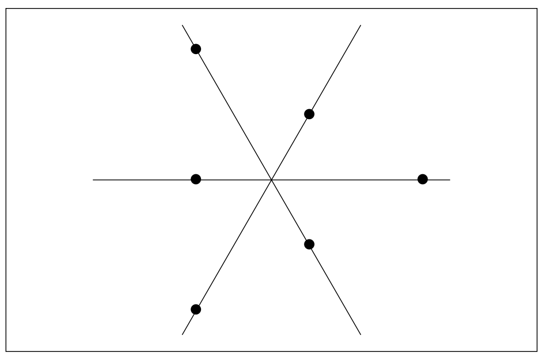
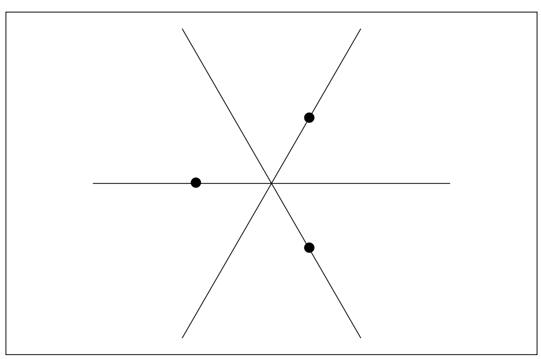


FIGURE 5. Roots of V_{2L_1}.



and $\wedge$ has these



One way to see these pictures is that since order "doesn't matter" in Sym^2, the double multiplicity won't show up, and therefore the remaining three are left to $\wedge^2$. Another way to see this is that the third exterior power is trivial, so

$$V_{L_1}^* \simeq \wedge^2 V_{L_1}$$

In the end we get:

$$V_{2L_1} = \mathrm{Sym}^2\left(V_{L_1}\right)$$

4. A step back

Now we want to generalize this story. To move towards this, we compile a list of "theoretical ingredients" which will be a part of the general story:

(1) $\mathfrak{g}/\mathbb{C}$ simple Lie algebra (could generalize to semi-simple, reductive)
(2) $\mathfrak{h} \subseteq \mathfrak{g}$ Cartan subalgebra (not unique)
(3) $\mathfrak{h}^*$ is a quotient of $\mathfrak{g}^*$, which is the dual space of eigenvalues/weights.
(4) The non-zero eigenvalues of the adjoint representation of $\mathfrak{g}$ form the roots $R \subset \mathfrak{h}^*$.
(5) $\mathfrak{b} \subseteq \mathfrak{g}$ a Borel subalgebra (not unique) (contains $\mathfrak{h}$)
(6) $R^+ \subset R$ positive roots (roots inside Borel)
(7) $\Delta^+ \subset R^+$ simple roots ($R^+ \subset \mathbb{Z}_{\geq 0}\Delta^+$, $R \subset \mathbb{Z}\Delta^+$, Δ^+ linearly independent)
(8) $\Lambda^+ \subset \mathfrak{h}^*$ is the cone of integral dominant weights. (Also the highest weights with respect to $\mathfrak{b}$ for irreducible representations)

Schur functors next time and PBW for $\mathfrak{sl}(3, \mathbb{C})$. We will also discuss how to generalize this whole story, but will likely not prove it in detail.

LECTURE 14
MATH 261A

LECTURES BY: PROFESSOR DAVID NADLER
NOTES BY: JACKSON VAN DYKE

1. Recall

Recall we have the space of all weights $\mathfrak{h}^*$ which contains the dominant integral weights $\mathbb{Z}_{\geq 0}\langle L_1, -L_3\rangle$. This consists of non-negative integral multiples of the fundamental weights. Recall the fundamental weights are the highest weights for the standard representation and the standard dual representation.

Also recall we had the theorem:

Theorem 1. *Irreducible representations are in bijection with the dominant weights. In particular, we send an irreducible representation to its $\mathfrak{b}$ highest weight.*

Date: October 11, 2018.

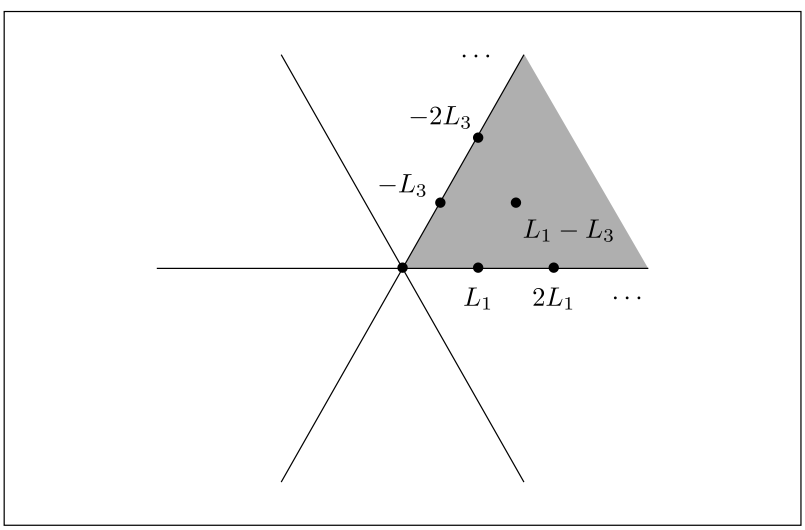


FIGURE 1. Dominant weights for $\mathfrak{sl}(3,\mathbb{C})$ if we take $\mathfrak{b}$ to be generated by the H_{ij} and X_{ij}.

2. What choices have we made so far

Professor Nadler says it's relatively fair to say that representation theory is the study of choices. So we now review some of the choices we have made so far in our discussion of $\mathfrak{sl}(3, \mathbb{C})$.

2.1. Cartan subalgebra.

Recall we chose a Cartan subalgebra $\mathfrak{h}$ to be some maximal abelian subalgebra. This is not a unique choice, but we do have the following:

Proposition 1. *Let G have Lie algebra $\mathfrak{g}$.*

1. *All Cartan subalgebras $\mathfrak{h} \subseteq \mathfrak{g}$ are conjugate by G under $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$.*
2. *The* Weyl group
$$W_{\mathfrak{g}} = N_G(\mathfrak{h}) / Z_G(\mathfrak{h}) = N_G(\mathfrak{h}) / H$$
is a finite, where $H \subseteq G$ is the subgroup with Lie algebra $\mathfrak{h}$.

Remark 1. We don't need to take the unique simply-connected G since whether or not we quotient out by the center $Z(G)$ won't affect the action Ad, so it won't change whether or not these things are related by conjugation.

Remark 2. It is very beautiful when the action of a group is transitive, since it is somehow enough to only understand the action on one element. But then we have to ask another very important question, which is what the stabilizer of this one is, and that's what led us to the second half of this proposition.

Remark 3. The ambiguity of making a certain choice of Cartan subalgebra is somehow recorded by the $W_{\mathfrak{g}}$ action on $\mathfrak{g}$ by conjugation.

Example 1. We now calculate the Weyl group for $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$. In this case
$$W_{\mathfrak{g}} = \Sigma_n$$
The action of this on $\mathfrak{h} \subseteq \mathfrak{g}$, i.e. the traceless diagonal matrices, is called the standard representation of $W_{\mathfrak{g}}$.

The action is explicitly given by permutation matrices. For example under $\sigma = (12)$,
$$\begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 1 & \\ -1 & 0 & \\ & & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix} \begin{pmatrix} 0 & -1 & \\ 1 & 0 & \\ & & 1 \end{pmatrix} = \begin{pmatrix} \lambda_2 & & \\ & \lambda_1 & \\ & & \lambda_3 \end{pmatrix}$$
So this acts on the space of eigenvalues by permuting them as expected.

In the language of the diagrams we have been drawing, the three lines that we were just sort of using to orient ourselves are really representing the hyperplanes over which the elements of $W_{\mathfrak{g}}$ are reflecting. For example, $\sigma = (12)$ is reflecting across the $-L_3$ line.

2.2. Borel subalgebra.

We also saw that we have to choose a Borel subalgebra inside of $\mathfrak{g}$ which contains $\mathfrak{h}$. We have a similar proposition for this choice:

Proposition 2. *Let G have Lie algebra $\mathfrak{g}$.*

1. *All Borel subalgebras are related by conjugation under $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$.*
2. *The stabilizer of any $\mathfrak{b}$ is the subgroup $B \subseteq G$ with Lie algebra $\mathfrak{b}$.*

Definition 1. The *flag variety* $\mathcal{B}$ of $\mathfrak{g}$ is the space of Borel subalgebras.

The proposition tells us that the flag variety is just G/B since G acts transitively on this space, and the stabilizer is B.

Remark 4. The flag variety is the space of choices for Borel subalgebras.

Example 2. For $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, we calculate the flag variety. So consider all of the Borel subalgebras inside $\mathfrak{g}$. This is just an ambient vector space, and each Borel subalgebra is a subspace, so we should think of this flag variety as being a submanifold of the Grassmannian of subspaces of $\mathfrak{g}$ i.e.

$$\mathcal{B} \subseteq \mathrm{Gr}(\dim \mathfrak{b}, \dim \mathfrak{g})$$

This all seems a bit abstract, but we're just looking for like k planes in l space, and then some of these are Borel subalgebras and that's what we want.

So let $G = \mathrm{SL}(n, \mathbb{C})$, and B be the upper triangular matrices in $\mathrm{SL}(n, \mathbb{C})$.

Claim 1. G/B is naturally isomorphic to flags

$$\langle 0 \rangle = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_n = \mathbb{C}^n$$

where $\dim E_i = i$.

Exercise 1. Prove this claim. That is, show that $\mathfrak{sl}(n)$ acts transitively on flags, and that $\mathfrak{b}$ is the stabilizer of the standard flag where $E_i = \mathrm{Span}\langle e_1, \cdots, e_i \rangle$.

Solution. This is somehow a standard exercise in linear algebra. Show every flag can be split into a basis, and then $\mathfrak{sl}(n)$ acts transitively on the basis.

Example 3. For $n = 1$ the flag variety is a point. For $n = 2$, we are studying flags in $\mathbb{C}^2$. Since the ends are fixed, every flag is just a choice of lines, which is just $\mathbb{CP}^1$.

For $n = 3$, this consists of lines E_1 inside planes E_2, inside $\mathbb{C}^3$. The collection of these isn't anything special we have seen before, but it is inside the collection of choices of lines crossed with choices of planes:

$$\mathbb{CP}^2 \times \left(\mathbb{CP}^2\right)^*$$

The line is represented by a vector v, and the plane is represented by a covector w. The condition is just that the line must be inside the plane. In particular, E_1 is the span of v and E_2 is orthogonal to w, the kernel of w. So the flag variety is cut out by the equation $w(v) = 0$. So we start with four dimensions, and insisting on this equation gives us three dimensions, which is good since $\mathfrak{sl}(3)$ is eight-dimensional, and B is 5-dimensional.

There are many beautiful things to be said about flag varieties, but we just state one more thing. Recall we really liked $\mathbb{CP}^1$ since it was just projective space. But as it turns out, we can think of this flag variety as a sort of iterated projective space. So let's say we forget the line and remember the hyperplane, so we're projecting:

$$\begin{array}{c} \mathrm{SL}(n, \mathbb{C})/B(n) \\ \big\downarrow \\ \left(\mathbb{CP}^{n-1}\right)^* \end{array}$$

and then the rest of the data for this fixed hyperplane is just a flag in this hyperplane, so the fiber is $\mathrm{SL}(n-1, \mathbb{C})/B(n-1)$.

The $n = 2$ example was literally $\mathbb{CP}^1$, the $n = 2$ example was just sort of roughly a $\mathbb{CP}^1$ and a $\mathbb{CP}^2$, and the next one is put together as a $\mathbb{CP}^1$, $\mathbb{CP}^2$, and a $\mathbb{CP}^3$. This is however not to say that these aren't somehow put together in an interesting way, because they are.

2.3. **Borus.** The two choices of a Cartan and Borel subalgebra together make the choice of a Borus, and now we have the following proposition bringing them together:

Proposition 3. *Let G be a Lie group with Lie algebra $\mathfrak{g}$, then*

1. *All "boruses" are conjugate by G.*
2. *The stabilizer is isomorphic to $Z_G(\mathfrak{h}) \simeq H$.*

2.4. **Back to representation theory.** Now we have the following as a result of these propositions:

Corollary 1. *The Weyl group acts simply transitively on the Borel subalgebras containing $\mathfrak{h}$.*

Exercise 2. Prove that the above three propositions imply this corollary.

So note that the choice of a borus that we made last time determined which chamber was the dominant one. There are actually four other choices of chambers that are just as good. We illustrate this with some examples. For all of them fix $\mathfrak{h} \subseteq \mathfrak{sl}(n, \mathbb{C})$ the usual diagonal Cartan subalgebra.

Example 4. Let $n = 2$. Then $\Sigma_2 \circlearrowright \langle \mathfrak{b} \text{ containing } \mathfrak{h} \rangle$ acts simply transitively.

$$\mathfrak{b} = \left\langle \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\rangle \quad \mapsto \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathfrak{b} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \left\langle \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \right\rangle = \mathfrak{b}^{\mathrm{op}}$$

Example 5. Let $n = 3$. Then $\Sigma_3 \cdot \langle \mathfrak{b} \text{ containing } \mathfrak{h} \rangle$. We know $\mathfrak{b}$ upper triangular matrices works. Now we can conjugate to find the others. We know $\Sigma_2 = \langle (12), (23) \rangle$. first we lift these to matrices:

$$(12) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (23) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

FIGURE 2. These are two distinct tangles which represent products of transpositions which are the same in Σ_3.

Now we conjugate to get:

$$\langle e_1\rangle \subset \langle e_1, e_2\rangle \qquad \langle e_2\rangle \subset \langle e_1, e_2\rangle \qquad \langle e_3\rangle \subset \langle e_1, e_3\rangle \qquad \langle e_2\rangle \subset \langle e_1, e_2\rangle$$

$$\begin{tikzcd}
& \begin{pmatrix} * & 0 & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \arrow[r, "(23)"] & \begin{pmatrix} * & * & 0 \\ 0 & * & 0 \\ * & * & * \end{pmatrix} \arrow[dr, "(12)"] & \\
\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \arrow[ur, "(12)"] \arrow[dr, "(23)"] & & & \begin{pmatrix} * & 0 & 0 \\ * & * & 0 \\ * & * & * \end{pmatrix} \\
& \begin{pmatrix} * & * & * \\ 0 & * & 0 \\ 0 & * & * \end{pmatrix} \arrow[r, "(12)"] & \begin{pmatrix} * & 0 & 0 \\ * & * & * \\ * & 0 & * \end{pmatrix} \arrow[ur, "(23)"] &
\end{tikzcd}$$

$$\langle e_1\rangle \subset \langle e_1, e_2\rangle \qquad \langle e_1\rangle \subset \langle e_1, e_3\rangle \qquad \langle e_2\rangle \subset \langle e_2, e_3\rangle \qquad \langle e_2\rangle \subset \langle e_1, e_2\rangle$$

Exercise 3. Show that this is the case by explicitly conjugating.

We write the flags stabilized by these choices of Borels above and below the diagram. The fact that these paths give the same final result is a result of the fact in fig. 2.

Now fix a Cartan subalgebra $\mathfrak{h}$. Inside of $\mathfrak{h}^*$ we want to talk about the $\mathfrak{b}$-dominant integral weights $\Lambda^+ \subseteq \mathfrak{h}^*$. All integral weights form a lattice inside $\mathfrak{h}^*$, and now we can ask how we chose this cone. We chose those which were "positive" with respect to $\mathfrak{b}$. The cone Λ^+ consisted of the possible highest weights for $\mathfrak{b}$. So if we conjugate $\mathfrak{b}$ by a permutation matrix, this is just acting this permutation on this cone by reflecting over the L_i, so it gives us the alternative cones.

3. Construction of irreducible representations

Fix a borus $\mathfrak{h} \subset \mathfrak{b}$. We want to construct the irreducible representation with a given highest weight. Recall in the $\mathfrak{sl}(2, \mathbb{C})$ case we saw

$$V_n = \operatorname{Sym}^2 V_1$$

and

$$\bigoplus_{n=0}^{\infty} V_n = \mathbb{C}[u, v]$$

so since we're trying to construct polynomials we might have guessed that.

So now in the $\mathfrak{sl}(3, \mathbb{C})$ case, we have $V_0 = \mathbb{C}$ is trivial, $V_{L_1} \simeq \mathbb{C}^3$ is the standard representation, and $V_{-L_3} \simeq \mathbb{C}^3$ is the dual standard representation. Then we claim the following:

Claim 2. $\operatorname{Sym}^n(V_{L_1})$ is irreducible with highest weight nL_1, and $\operatorname{Sym}^n(V_{-L_3})$ is irreducible with highest weight $-nL_3$.

So we have the same $\mathfrak{sl}(2, \mathbb{C})$ picture along the $-L_3$ and L_1 lines. Just as before we have the following decomposition:

$$\bigoplus_n \operatorname{Sym}^n V_{L_1} \simeq \mathbb{C}[u, v, w] \qquad \bigoplus_n \operatorname{Sym}^n V_{-L_3} \simeq \mathbb{C}[u^*, v^*, w^*]$$

LECTURE 15
MATH 261A

LECTURES BY: PROFESSOR DAVID NADLER
NOTES BY: JACKSON VAN DYKE

1. Clarifications

We will continue our discussion of representations of $\mathfrak{sl}(3,\mathbb{C})$. But first some clarifications. We saw $\Lambda = \mathbb{Z}\langle L_1, L_2, L_3\rangle$ and $\Lambda^+ = \mathbb{Z}_{\geq 0}\langle L_1, -L_3\rangle$ concretely, but now we offer a sort of invariant definition.

1.1. **Weight lattice.** Any time we have $\mathfrak{g}/\mathbb{C}$ a Lie algebra, we can associate to this a simply-connected complex Lie group $G/\mathbb{C}$. For example, for $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{C})$ we get $G = \mathrm{SL}(n,\mathbb{C})$. Now if we choose a borus in $\mathfrak{g}$, we will get subgroups of G that play a similar role. So choosing $\mathfrak{h} \subseteq \mathfrak{b} \subseteq \mathfrak{g}$ we will get subgroups $H \subseteq B \subseteq G$.

Now we want to compare $\mathfrak{h}^* = \mathrm{Hom}_{\mathbf{Vect}}(\mathfrak{h},\mathbb{C})$ to $\Lambda = \mathrm{Hom}_{\mathbf{Ab}}(H,\mathbb{C}^\times)$. We know $\mathfrak{h}^*$ consists of the eigenvalues of the (irreducible) $\mathfrak{h}$ representations, and Λ consists of the eigenvalues of the irreducible H-representations. This tells us that $\mathfrak{h}^* \simeq \mathbb{C}^{\dim\mathfrak{h}}$ and $\Lambda \simeq \mathbb{Z}^{\dim H}$. Note also that $\mathfrak{h} \simeq \mathbb{C}^{\dim\mathfrak{h}}$ and $H \simeq (\mathbb{C}^\times)^{\dim H}$.

In fact, we naturally have that the weight lattice Λ is contained in $\mathfrak{h}^*$. This map is differentiation, since Λ consists of maps of Lie groups, and $\mathfrak{h}^*$ consists of maps of Lie algebras. But this is not equality, since there are plenty such maps of Lie algebras that don't come from maps of Lie groups.

Warning 1. This inclusion is proper since H is not simply-connected.

1.2. **Dominant weight lattice.** Recall the roots R are the nonzero eigenvalues of the adjoint representation ad. Half of these will be in our choice of Borel subalgebra. We call these roots the positive roots $\Delta^+ \subseteq R$. Equivalently these are the roots in the adjoint representation of $\mathfrak{b}$. Then there are the simple roots $\Sigma^+ \subseteq \Delta^+$ which form a basis. Now write R^+ for the positive root cone $R^+ = \mathbb{Z}_{\geq 0}\Delta^+$.

Recall the killing form is an inner product on $\mathfrak{g}$.

Exercise 1. Show that $\mathfrak{g}$ is semisimple iff the killing form is nondegenerate. This is called Cartan's criterion.

This means it induces an inner product on $\mathfrak{g}^*$, and in particular on $\mathfrak{h}^*$ by restriction. Then the dominant cone consists of the lattice points which are non-negative when paired with the positive root cone. Explicitly:

$$\Lambda^+ = \left\{\lambda \in \Lambda \subseteq \mathfrak{h}^* \,|\, \forall \alpha \in \Delta^+, \langle\lambda,\alpha\rangle \geq 0\right\}$$

Date: October 16, 2018.

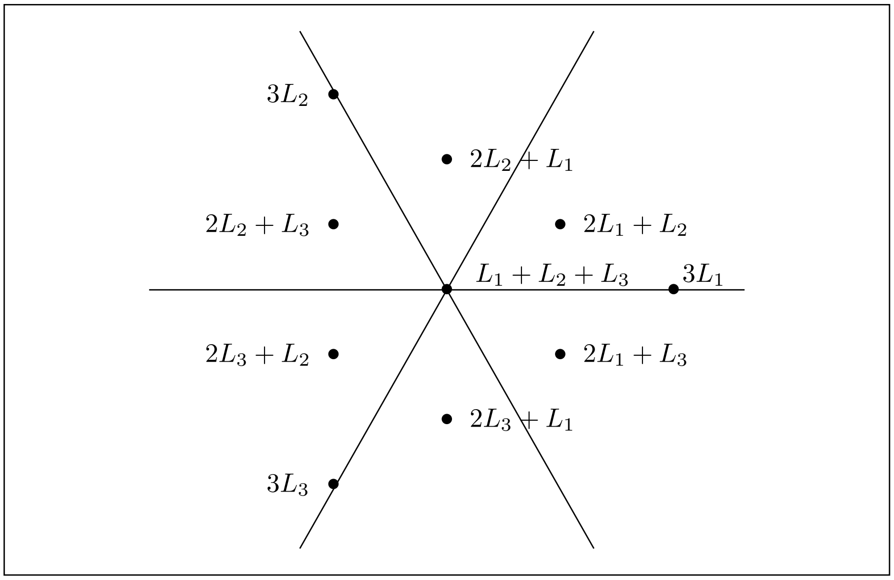


FIGURE 1. Roots of $V_{3L_1} = \mathrm{Sym}^3 V_{L_1}$.

2. Constructing irreducible representations of $\mathfrak{sl}(3, \mathbb{C})$

We want to construct irreducible representations V_λ for $\lambda \in \Lambda^+$ for $\mathfrak{sl}(3, \mathbb{C})$. Recall we have already seen $V_{L_1} = \mathbb{C}^3$ is the standard representation and $V_{-L_3} = V_{L_1+L_2}$. When this is written in the first way it is supposed to be dual to the standard, and the second way suggests it is $\wedge^2\mathbb{C}^3$. We also saw that $V_{\alpha_{13}}$ was the adjoint representation. Finally we saw that we have $\mathrm{Sym}^n V_{L_1}$ has highest weight nL_1 and the other weights are as in the following example.

Example 1. Consider $\mathrm{Sym}^3 V_{L_1}$. This has weights as in fig. 1.

A similar story holds for $-L_3$.

Example 2. Consider $\mathrm{Sym}^2 V_{-L_3} = V_{-2L_3}$. This has weights as in fig. 2.

The question that remains, is what if we want a representation which is a linear combination of L_1 and L_2 such as $mL_1 - nL_3$. The idea here is that this highest lives in $\mathfrak{h}^*$ as in fig. 3 and then we claim the following:

Claim 1. The non-zero weights of V_λ lie in the convex hull of $W \cdot \lambda$ as in fig. 3.

Remark 1. Notice that in the case of V_{nL_1} and V_{-mL_3} we have a sort of degenerate hexagon.

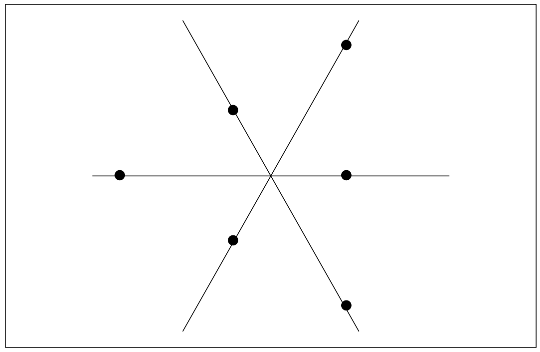


FIGURE 2. Weights of V_{-2L_3}.

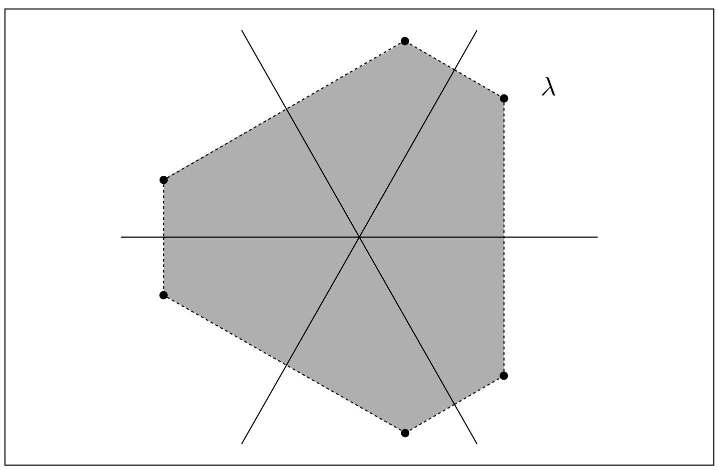


FIGURE 3. The convex hull $W \cdot \lambda$.

The idea of this proof will be to restrict to the copies of $\mathfrak{sl}(2,\mathbb{C})$ inside $\mathfrak{sl}(3,\mathbb{C})$. In particular the block diagonal matrices:

$$\mathfrak{l}_{12} = \begin{pmatrix} * & * & \\ * & * & \\ & & \end{pmatrix} \qquad \mathfrak{l}_{23} = \begin{pmatrix} & & \\ & * & * \\ & * & * \end{pmatrix} \qquad \mathfrak{l}_{13} = \begin{pmatrix} * & & * \\ & & \\ * & & * \end{pmatrix}$$

These are examples of *Levi subalgebras*. The roots in $\mathfrak{l}_{12}$ are as in fig. 4.

We want to think of these subalgebras as moving along the line spanned by their roots in the same sense that $\mathfrak{sl}(2,\mathbb{C})$ moved along the real line[1]. For example if we restrict a representation of $\mathfrak{sl}(3,\mathbb{C})$ to (say) $\mathfrak{l}_{12}$. We get these lines running diagonal all parallel to the line connected α_{12} and α_{21}.

[1] Professor Nadler says the secrets to the universe come from understanding the Cartan subalgebra, and understanding $\mathfrak{sl}(2)$.

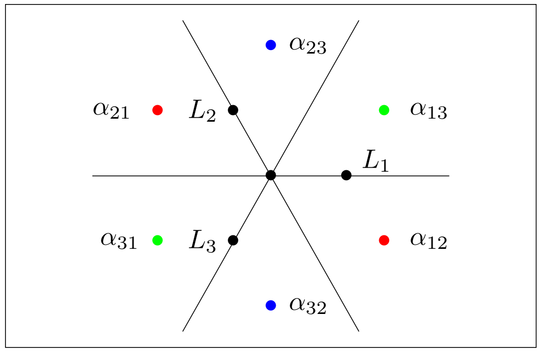


FIGURE 4. In red we have the roots of $\mathfrak{l}_{12}$, in green we have the roots of $\mathfrak{l}_{13}$, and in blue we have the roots of $\mathfrak{l}_{23}$.

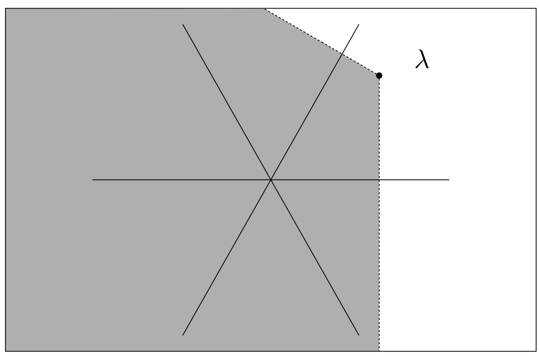


FIGURE 5. Because λ is the highest weight, and because all of the weights are given by acting Y on the highest weight, we know that every weight must be contained in this hull.

Proof. Say we have some highest weight, then the Xs all bring it to zero. We haven't shown that repeatedly acting Y on the highest weight gives us everything, but taking that for granted, nothing is nonzero outside of the hull pictured in fig. 5. But now it must be symmetric about the lines which W reflects over since all of these parallel lines are $\mathfrak{sl}(2, \mathbb{C})$ representations. This is exactly the convex hull in fig. 3. $\square$

3. Constructing irreducible representations of $\mathfrak{sl}(3, \mathbb{C})$

Recall we saw:

$$\mathfrak{sl}(2, \mathbb{C}) \circlearrowright \mathbb{C}^2 \rightsquigarrow \bigoplus_{n=0}^{\infty} \mathrm{Sym}^n \left(\mathbb{C}^2\right) = \mathcal{O}\left(\mathbb{C}^2\right)$$

where $\mathcal{O}\left(\mathbb{C}^2\right)$ consists of polynomial functions. Now we have $\mathfrak{sl}(3, \mathbb{C}) \circlearrowright \mathbb{C}^3$ standard, as well as the dual to this and we want to build a similar picture.

3.1. **Fundamental affine space.** The fundamental affine space is:

$$X_{\mathfrak{sl}(3,\mathbb{C})} = \left\{(v,\lambda) \in \mathbb{C}^3 \times (\mathbb{C}^*)^3 \mid \lambda(v) = 0\right\}$$

Another, more general, way of thinking about this is:

$$X_{\mathfrak{sl}(3,\mathbb{C})} = \left\{(v, w_1 \wedge w_2) \in \mathbb{C}^3 \times \wedge^2\mathbb{C}^3 \mid v \wedge (w_1 \wedge w_2) = 0\right\}$$

this is nice since it looks like a version of a flag.

Claim 2.

$$\mathcal{O}\left(X_{\mathfrak{sl}(3,\mathbb{C})}\right) = \bigoplus_{\lambda \in \Lambda^+} V_\lambda$$

where every irreducible appears exactly once.

So every simple Lie algebra has such a fundamental affine space, so this should give some hint as to how we should generalize this.

Remark 2. The proof next time will follow from the Peter-Weyl theorem.

Let's find some of our favorite representations in this.

Example 3. Write v and λ in coordinates:

$$v = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad \lambda = (y_1, y_2, y_3)$$

The trivial representation is given by constant functions, $V_{L_1} = \mathbb{C}^3 = \mathbb{C}\langle x_1, x_2, x_3\rangle$, and similarly $V_{-L_3} = \left(\mathbb{C}^3\right)^* = \mathbb{C}\langle y_1, y_2, y_3\rangle$. The adjoint representation is given by:

$$V_{L_1 - L_3} = \mathfrak{sl}(3,\mathbb{C}) = \mathbb{C}\langle x_i y_j\rangle$$

where these satisfy:

$$x_1y_1 + x_2y_2 + x_3y_3 = 0$$

so this object is 8-dimensional.

LECTURE 16

LECTURES BY: PROFESSOR DAVID NADLER
NOTES BY: JACKSON VAN DYKE

1. Fundamental affine space

Definition 1. The fundamental affine space X_n of $\mathfrak{sl}(n, \mathbb{C})$ is contained in

$$X_n \subset \mathbb{C}^n \times \wedge^2 \mathbb{C}^n \times \cdots \times \wedge^{n-1} \mathbb{C}^n$$

In particular, it comprises collections of the following form:

$$(a_1, b_1 \wedge b_2, c_1 \wedge c_2 \wedge c_3, \cdots)$$

These are elementary forms in the sense that they aren't sums of such elements. We also insist on the "inclusions" $a_1 \wedge (b_1 \wedge b_2) = 0$ and the higher-dimensional analogues[1] i.e. $a_1 \wedge (c_1 \wedge c_2 \wedge c_3)$ etc. I.e. the spans are included in the larger if nonzero.

This is supposed to look like the flag variety.

To see that this isn't so mysterious, consider the open subset $X_n^0 \subset X_n$ where all terms are nonzero. This space has a natural projection

$$\begin{array}{c} X_n^0 \\ \downarrow \\ \mathcal{B} \end{array}$$

to the flag variety of flags in n-space, $\mathcal{B}$, where we map these primitive forms to their span. This makes sense since we required them to be nonzero.

This is surjective, and in fact a fibration with fiber as follows. Each time we sort of introduce a new vector, all we care about is preserving the "volume" of the parallelepiped, there's sort of $\mathbb{C}^\times$ many choices. So the fibers are $(\mathbb{C}^\times)^{n-1}$.

Note the following:

$$\dim \mathcal{B}_n = \frac{n(n-1)}{2} \qquad \dim X_n^0 = \frac{n(n-1)}{2} + n - 1 = \frac{(n+2)(n-1)}{2}$$

Now we have the following lemma:

Lemma 1. *$X_0^n \simeq G/N$ where $N = [B, B] = \left\langle b_1 b_2 b_1^{-1} b_2^{-1} \in B \right\rangle$ consists of upper diagonal matrices with 1 on the diagonal.*

Proof. We need to show G acts transitively, and the stabilizer is N. I.e. we take any list of such forms to any other list of forms using G. So we write down our favorite element of X_0^n:

$$(e_1, e_1 \wedge e_2, \cdots, e_1 \wedge \cdots \wedge e_{n-1})$$

Date: October 18, 2018.

[1] These are called Plücker equations.

and then any other one is:

$$(a_1, b_1 \wedge b_2, c_1 \wedge c_2 \wedge c_3, \cdots)$$

and we need a matrix in $\mathrm{SL}(n)$ which takes us there. The first column should just be a_1. Now because of the inclusion equations, we can write $b_1 \wedge b_2$ as $a_1 \wedge b_2'$ for some b_2'. Explicitly, we can write a matrix with columns:

$$\begin{pmatrix} a_1 & b_2' & \cdots \end{pmatrix}$$

this is basically just a change of basis matrix.

Now what group elements fix this nested sequence? We better have $1, 0, \cdots, 0$ in the first column, and we have to maintain the span of b_1 and b_2, so we have to have 0 after the second coordinate, so we get $(*, 1, 0, \cdots, 0)$ in the second column, since we have 1 on the diagonal. All together we get:

$$\begin{pmatrix} 1 & * & \cdots & \cdots \\ 0 & 1 & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

which is of course N. □

Corollary 1. *Recall we already saw $\mathcal{B}_n = G/B$, so this is a fibration for $B/N \simeq T$.*

Example 1. The fundamental affine space of $\mathfrak{sl}(2, \mathbb{C})$ is $X_2 = \mathbb{C}^n$ and $X_2^0 = \mathbb{C}^2 \setminus \{0\}$. The projection maps $v \mapsto l \in B_n \simeq \mathbb{P}^1$ where $l \simeq \mathbb{C}\langle v \rangle$.

Remark 1 (For algebraic geometers). We have these two creatures X_n^0 and X_n and we might sort of wonder why we're considering both of them. Dealing with just G/N is very nice, but it is not an affine variety, as we saw in the previous example: $\mathbb{C}^2 \setminus 0$ is not affine (though it is quasi-affine). The affine closure of X_n^0 is X_n.

Example 2. We know the dimension of $\mathfrak{sl}(3, \mathbb{C})$ is 8, and then the dimension of X_3^0 is $8 - 3 = 5$. T is two dimensions so when we divide by this we get down to the three-dimensional flag variety. We start with $a_1 = ae_1$ then $b_2' = be_2$, and all together

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & a^{-1}b^{-1} \end{pmatrix}$$

since we need determinant one. So this is a map from points above to determinant one diagonal matrices.

The fiber living above the standard flag:

$$E_1 = \mathbb{C}\langle e_1 \rangle \qquad E_2 = \mathbb{C}\langle e_1, e_2 \rangle$$

is $(\mathbb{C}^\times)^2$, so it is somehow missing the axes. Then the closure is $\overline{T} \simeq \mathbb{C}^2$, but we got this only from paying attention to a and b, but we really want something which pays equal attention to all coordinates. More democratically, T is naturally a subset of $(\mathbb{C}^\times)^3$ cut out by $\det = 1$. The quotient picture was like a photograph of the corner of the room, and this is like the slice of the corner of the room.

1.1. **Relationship with fundamental representations.** We now return to the following proposition:

Proposition 1.

$$\mathcal{O}(X_n) = \bigoplus_{\lambda\in\Lambda^+} V_\lambda$$

where V_λ is a representation of highest weight λ, and each V_λ appears exactly once.

Proof. We need to calculate the highest weights in $\mathcal{O}(X_n)$. Recall these are the invariants under N, so they are in $\mathcal{O}(X_n)^N$ where $N = [B,B]$ consists of the strictly upper triangular matrices as usual. Recall the Lie algebra of N is $\mathbb{C}\langle X_{ij}\rangle$.

Claim 1. There exists an open B orbit in X_n isomorphic to B.

Proof. Take the opposite standard flag,

$$e^{\mathrm{op}} = \{e_n, e_n \wedge e_{n-1}, \cdots\}$$

and then we claim B acts on this with an open orbit. Start with e_3, then take $e_3 \wedge e_2$.

Exercise 1. Show that $B\cdot e^{\mathrm{op}}$ consists of all configurations with nonzero terms with spans transverse to the standard configuration e. Also check that if $b\cdot e^{\mathrm{op}} = e^{\mathrm{op}}$ then $b = 1$.

and we are done. □

This means $B \simeq B\cdot e^{\mathrm{op}} \subseteq X_n$ is an open dense subset. Now take the functions $\mathcal{O}(X_n)$ and restrict them to $\mathcal{O}(B\cdot e^{\mathrm{op}}) = \mathcal{O}(B)$, and since B is dense this must be an inclusion. Now we can also restrict:

$$\mathcal{O}(X_n)^N \hookrightarrow \mathcal{O}(B\cdot e^{\mathrm{op}})^N \simeq \mathcal{O}(N\backslash B) \simeq \mathcal{O}(T)$$

So N invariant functions give us functions on T. In conclusion we have an injection:

$$\mathcal{O}(X_n)^N \hookrightarrow \mathcal{O}(T)$$

But we know the weight lattice Λ is just monomial functions on T, $\Lambda = \mathrm{Hom}_{\mathbf{Ab}}(T, \mathbb{C}^\times)$, so $\mathcal{O}(T)$ is just the $\mathbb{C}$-span of the weight lattice $\mathbb{C}\langle\Lambda\rangle$.

Example 3. The idea here is

$$\mathcal{O}(\mathbb{C}^\times) = \left\{\sum_{i=-N}^{N} c_i z^i\right\}$$

and in this case $\Lambda \simeq \mathbb{Z} = \{z^i \mid i \in \mathbb{Z}\}$.

So to every N-invariant function we have assigned a linear combination of weights. But we know N-invariants are highest weights, so the actual function we get can't be arbitrary, it has to be highest weight. I.e. the image of any particular highest weight must be a monomial. I.e. the injection above is T-equivariant.

There is a G action $G \circlearrowright \mathcal{O}(X_n)$ where $(g\cdot f)(x) = f(g^{-1}x)$. Now look at $\mathcal{O}(X_n)^N$. Then claim that this still has a T action given by the same formula. So $(t\cdot f)(x) = f(t^{-1}x)$. So now we want to check the following. Look at $n(tf)$ and we want to show that this is just tf:

$$n(tf) = tt^{-1}(ntf) = t(n'f) = tf$$

so there's still a T action.

Since the construction is T-equivariant we have the map

$$\mathcal{O}\left(X_n\right)^N \hookrightarrow \mathcal{O}\left(T\right)$$

and a left T-action on both. This is just since restriction "commuted" with the T-action. A highest weight vector of highest-weight λ must get mapped to some scale of the highest weight monomial z^λ. Therefore we can conclude that the highest weight vectors inject into the possible weights. I.e. there exists at most one dimension of highest weight vector for any given weight.

Now conversely we claim that

$$V_{\mathrm{std}}, \wedge^2 V_{\mathrm{std}}, \cdots, \wedge^{n-1} V_{\mathrm{std}} = V_{\mathrm{std}}^*$$

are all inside $\mathcal{O}\left(X_n\right)$. Recall the highest weights of these representations are a basis for the dominant weights. Now let $f_1, \cdots, f_{n-1}$ be highest weight vectors in each of the $\wedge^i V_{\mathrm{std}}$ inside $\mathcal{O}\left(X_n\right)$. Products of these will be nonzero, and this product will still be N invariant. Therefore it has to contain at least one irreducible of every highest weight. I.e. $f_1^{i_1} f_2^{i_2} \cdots f_{n-1}^{i_{n-1}}$ is a nonzero N-invariant vector of weight $i_1\lambda_1 + \cdots + i_{n-1}\lambda_{n-1}$ i.e. a highest weight of this eigenvalue. This is in Λ^+, therefore for every $\lambda \in \Lambda^+$ there exists highest weight representation V_λ inside $\mathcal{O}\left(X_n\right)$. ■

LECTURE 17
MATH 261A

LECTURES BY: PROFESSOR DAVID NADLER
NOTES BY: JACKSON VAN DYKE

Today we will finish discussion representation of $\mathfrak{sl}(3,\mathbb{C})$ and then talk about representations of $\mathfrak{sp}(4)$.

1. Representations of $\mathfrak{sl}(3,\mathbb{C})$

1.1. **Recall.** Recall we have the following theorem:

Theorem 1. $\mathbf{Rep}_{fd}(\mathfrak{sl}(3,\mathbb{C}))$ *is semisimple and the irreducibles are in bijection with* $\lambda \in \Lambda^+$.

So far we have done the following:

1. Constructed $V_\lambda \subset \mathcal{O}(X_3)$ for $\lambda \in \Lambda^+$.
2. Some analysis of possible weights. For $\lambda \in \Lambda^+$ we discussed that the possible weights for any irreducible representation with this highest weight will live in some sort of generalize hexagon.
3. Semi-simplicity of this category. This is done in the exact same way as it was done for $\mathfrak{sl}(2,\mathbb{C})$. Recall we said that representations of $\mathfrak{sl}(2,\mathbb{C})$ are the same as representations of $\mathrm{SL}(2,\mathbb{C})$, which are the same as representations of $\mathrm{SU}(2)$, and then we put metrics on everything and decomposed. Similarly, we have:
$$\mathbf{Rep}_{\mathrm{fd}}(\mathfrak{sl}(3,\mathbb{C})) \simeq \mathbf{Rep}_{\mathrm{fd}}(\mathrm{SL}(3,\mathbb{C})) \simeq \mathbf{Rep}_{\mathrm{fd}}(\mathrm{SU}(3))$$
and these all have invariant inner products, so subrepresentations have orthogonal complements. This technique generalizes even further to all simple Lie algebras. All we're really doing here is bring it to its unique simply connected Lie group, then go to the maximal compact subgroup, and then construct an invariant metric.

What we haven't shown is that any two irreducibles with the same highest-weight must be isomorphic. Once we do this, we will be done with the proof of the theorem. Recall for $\mathfrak{sl}(2,\mathbb{C})$ this was accomplished using Verma modules, which we will use in this case as well.

1.2. **Verma modules.** Let $\mathfrak{g}$ be a simple Lie algebra, and $\mathfrak{h} \subseteq \mathfrak{b} \subseteq \mathfrak{g}$ be a choice of Borus. Recall $\mathfrak{b}$ is maximal solvable, and $\mathfrak{h}$ is maximal abelian, and ad-diagonalizable.

Note that $\mathfrak{h} \hookrightarrow \mathfrak{b} \twoheadrightarrow \mathfrak{b}/[\mathfrak{b},\mathfrak{b}]$, and in fact this composition is an isomorphism.[1] I.e. $\mathfrak{h}$ lives in $\mathfrak{b}$ as a subalgebra and a quotient. This tells us that $\mathfrak{b} \simeq \mathfrak{h} \ltimes [\mathfrak{b},\mathfrak{b}]$.

Date: October 23, 2018.

[1] It makes sense that the quotient $\mathfrak{b}/[\mathfrak{b},\mathfrak{b}]$ is abelian since $\mathfrak{b}$ is solvable.

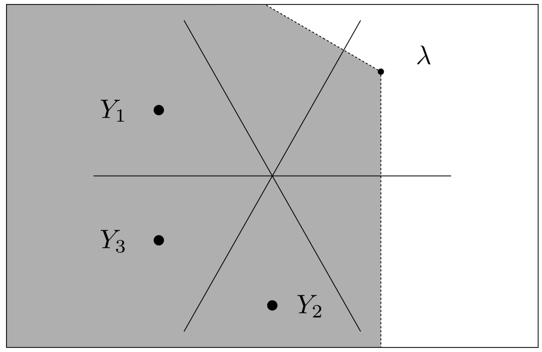


FIGURE 1. The ordering of the negative roots used in the calculations.

Fix a character (i.e. a linear map) $\lambda : \mathfrak{h} \to \mathbb{C}$ of $\mathfrak{h}$. In other words think of $\lambda \in \mathfrak{h}^*$ as a weight. We want to view λ as a 1-dimensional representation $\mathbb{C}_\lambda$ of $\mathfrak{h}$, i.e. C_λ is a complex line where $H \in \mathfrak{h}$ acts as: $H \cdot v = \lambda(H) v$.

Definition 1. The *Verma module* I_λ is

$$I_\lambda \coloneqq \mathcal{U}\mathfrak{g} \otimes_{\mathcal{U}\mathfrak{b}} \mathbb{C}_\lambda$$

Note that $\mathbb{C}_\lambda$ is a representation of $\mathfrak{b}$ via $\mathfrak{b} \to \mathfrak{h}$, so it can of course be a representation of $\mathcal{U}_\mathfrak{b}$. In other words, $X \cdot v = 0$ and $H \cdot v = \lambda(H) v$ for $X \in [\mathfrak{b}, \mathfrak{b}]$, and $H \in \mathfrak{h}$.

One might wonder why anyone would bother defining this in the first place. As it turns out, this is the standard way to construct a module which has the following universal property. If you find a vector in your representation such that X kills it, and H acts by λ, then there exists a unique map from I_λ to your representation. I.e.

$$\mathrm{Hom}_\mathfrak{g}(I_\lambda, V) \simeq \mathrm{Hom}_\mathfrak{g}(\mathbb{C}_\lambda, V)$$

which consists of the $[\mathfrak{b}, \mathfrak{b}]$-invariants and $\mathfrak{h}$ eigenvectors of weight λ.

Now the following gives us a basis for the Verma module:

Theorem 2 (PBW)**.** *Choose an ordering of the positive roots R^+, which gives us an ordering of the negative roots. Then a basis of $\mathcal{U}\mathfrak{g}$ is given by ordered monomials $Y^a H^b X^c$.*

This immediately implies that a basis for I_λ is given by the ordered monomials Y^a. This implies that we understand the weights of I_λ. Order the negative roots as in fig. 1. Then we can calculate the dimension of the weight spaces of these weights as in fig. 2. The weight spaces which are given by successive actions of Y_1 or Y_2 are all of dimension 1. However if we act by $Y_1 Y_2$, this is the same as just acting Y_3, so this space has dimension 2. As it turns out, each shell consists exactly of spaces with the same dimension, and every time you venture one shell deeper the dimension increases by 1.

Example 1. The corner of the third shell can be reached by monomials Y_3^2, $Y_1^2 Y_2^2$, and $Y_1 Y_2 Y_3$.

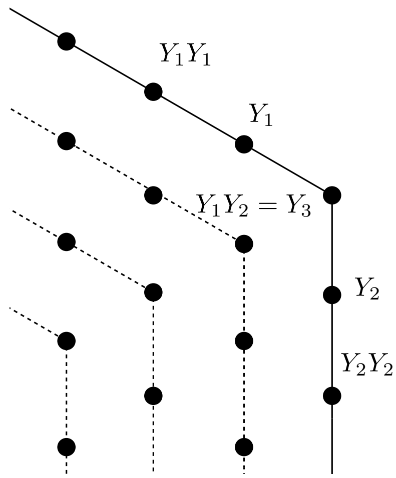


FIGURE 2. The different potential ways of reaching a given weight with ordered monomials gives the dimension of the weight space by PBW.

Exercise 1. Recall the pattern for $\mathfrak{sl}(2,\mathbb{C})$ was somehow linear with slope 0, then this is linear of slope 1. Find the pattern for $\mathfrak{sl}(4,\mathbb{C})$.

Now we want to write a closed formula for the character of I_λ. Well we know that it will somehow be $e_\lambda * (\cdots)$ for something inside. In particular:

$$\mathrm{ch}\,(I_\lambda) = e_\lambda * \left(\prod_{\alpha_i \in -R^+} \frac{1}{1-e_{\alpha_i}} \right)$$

We would love this to be a function with compact support on Λ, i.e. an element of $\mathbb{C}[\Lambda]$ for $\lambda \in \Lambda$, but we end up taking the completion $\widehat{\mathbb{C}[\Lambda]}$.

1.3. **Back to finite dimensional representations.** Suppose V is an irreducible finite dimensional representation of highest weight $\lambda \in \Lambda^+$. Now we will try to state some facts and arguments which will hopefully show that V is unique up to isomorphism.

Proposition 1. *The natural map $I_\lambda \to V$ given by the highest-weight vector is surjective.*

Proof. This is somehow a tautology, because if it wasn't surjective then the image would be a subspace of V, which is of course impossible since V is irreducible. $\square$

Remark 1. This is saying that we can get to anything in V by applying the Ys.

Now we construct a resolution of V in terms of the Verma modules. Return to $\mathfrak{sl}(3,\mathbb{C})$. We will discuss how it generalizes later. Recall we already know the nonzero weights of some representation with highest-weight λ lie in this sort of generalized hexagon. Then we can consider the first points where the Ys act as 0. To do this we define the following:

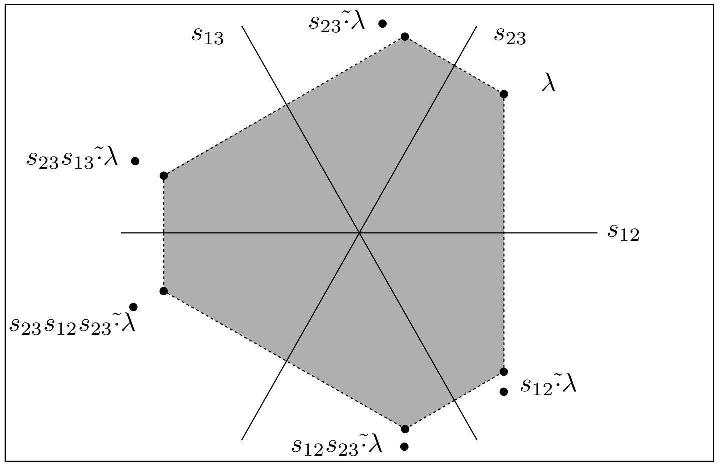


FIGURE 3. The images of λ under the $s_{ij}\tilde{\cdot}$ action.

Definition 2. Let $2\rho = \alpha_{12} + \alpha_{23} + \alpha_{13}$. Then define $s\tilde{\cdot}\lambda$ for $s \in W$ and $\lambda \in \mathfrak{h}^*$ to be the "reflection" with respect to hyperplanes translated by $-\rho$.

Remark 2. The point here is somehow that $-\rho$ was supposed to be the center of the universe all along rather than 0.

Example 2. We write the twisted product explicitly for single group actions:

$$s_{12}\tilde{\cdot}\lambda = s_{12}\lambda - \alpha_{12} \qquad s_{23}\tilde{\cdot}\lambda = s_{23}\lambda - \alpha_{23} \qquad s_{13}\tilde{\cdot}\lambda = s_{13}\lambda - \alpha_{13}$$

This explicitly tells us that $s_{12}\tilde{\cdot}\lambda$ and $s_{23}\tilde{\cdot}\lambda$ are the first time we escape the generalized hexagon from applying Ys as is evident in fig. 3.

This means that we have the following exact sequence:

$$I_{s_{12}\tilde{\cdot}\lambda} \oplus I_{s_{23}\tilde{\cdot}\lambda} \longrightarrow I_\lambda \longrightarrow\!\!\!\!\rightarrow V$$

but this isn't short exact since we still haven't somehow killed everything. So we keep considering the kernels to get the full resolution:

$$0 \longrightarrow I_{s_{12}s_{23}s_{12}\tilde{\cdot}\lambda} \longrightarrow I_{s_{12}s_{23}\tilde{\cdot}\lambda} \oplus I_{s_{23}s_{12}\tilde{\cdot}\lambda} \longrightarrow I_{s_{12}\tilde{\cdot}\lambda} \oplus I_{s_{23}\tilde{\cdot}\lambda} \longrightarrow I_\lambda \twoheadrightarrow V$$

Example 3. For $\mathfrak{sl}(2,\mathbb{C})$ the eigenvalue for upper-triangular matrices was 2, so $\rho = 1$, and then reflection about -1 is what gave us the term I_{-n-2} as the kernel in the SES.

Now we are somehow done, because here we just learned that for any irreducible, we are able to resolve it in terms of Verma modules. I.e. the part of the sequence without V has nothing to do with V. This is somehow just taking the maximal proper submodule of an object and repeating the process until we get 0.

1.4. **Weyl character formula.** The resolution from above also gives us the Weyl character formula since the character of V is the alternating sum of the preceding objects in the sequence.

Corollary 1 (Weyl character formula)**.**

$$\begin{aligned}\mathrm{ch}\,(V) &= \sum \mathrm{ch}\,(I_\lambda) - \left(\mathrm{ch}\,(I_{s_{12}\tilde{\cdot}\lambda}) + \mathrm{ch}\,(I_{s_{23}\tilde{\cdot}\lambda})\right) + \cdots \\ &= \sum_{w\in W} (-1)^{l(w)} \left(e_{w\tilde{\cdot}\lambda} * \prod_{\alpha_i \in R^+} \frac{1}{1-e_{\alpha_i}} \right)\end{aligned}$$

where l is the length of w.

Example 4. The length $l\,(w)$ in the case of $\mathfrak{sl}\,(3)$ is the number of these simple transpositions to get to w.

We will return to this next time when we will learn why this resolution is true, and that we can view it as some sort of algebraic realization of Schubert decomposition of the flag variety.

LECTURE 18

LECTURES BY: PROFESSOR DAVID NADLER
NOTES BY: JACKSON VAN DYKE

We will continue our discussion of Bernstein-Gelfand-Gelfand (BGG) resolution, and discuss $\mathfrak{sl}(4,\mathbb{C})$ and $\mathfrak{sp}(4,\mathbb{C})$ as generalizations of $\mathfrak{sl}(3,\mathbb{C})$.

1. BGG resolution

Informally speaking this is a resolution of a finite dimensional representation theory by Verma modules. More specifically, if we fix some highest-weight $\lambda \in \Lambda^+$, then we define:

$$\rho = \frac{1}{2} \sum_{\alpha_i \in R^+} \alpha_i$$

and modify the action of the Weyl group to act as:

$$s \tilde{\cdot} \lambda = s(\lambda + \rho) - \rho$$

for $s \in W$. So ρ is now the center of the universe. Now we choose some simple reflections, e.g. for $\mathfrak{sl}(3,\mathbb{C})$, s_{12} and s_{23} generate the group so we set these to be our simple reflections. Then the last ingredient is a length function $l : W \to \mathbb{Z}_{\geq 0}$ where $l(s)$ is the minimum word length of s in terms of simple reflections. Then this gives us the resolution:

$$0 \leftarrow V_\lambda \leftarrow I_\lambda \leftarrow \bigoplus_{l(s)=1} I_{s\tilde{\cdot}\lambda} \leftarrow \cdots \leftarrow I_{w_0\tilde{\cdot}\lambda} \leftarrow 0$$

At each stage if we ask if it's injective the answer is no[1] since there will be a kernel, and in particular it will be the maximal submodule.

One might be worried about the choice of these reflections depending on the fact that the Weyl group is S_n for $\mathfrak{sl}(n,\mathbb{C})$, but the point is, for every simple root, there will be a Levi $\mathfrak{sl}(2,\mathbb{C})$ living inside the Lie algebra, with that simple root as its positive root, and the reflection for $\mathfrak{sl}(2,\mathbb{C})$ will be a simple reflection for the Lie algebra.

2. Weights and roots of $\mathfrak{sl}(4,\mathbb{C})$

Consider the Lie algebra $\mathfrak{sl}(4,\mathbb{C})$. We proceed the same way as we did for $\mathfrak{sl}(3,\mathbb{C})$ and consider the weights of the standard representation on $\mathbb{C}^4$ as in fig. 1. Now just as α_{12} and α_{23} were simple roots for $\mathfrak{sl}(3,\mathbb{C})$, we now have three simple roots α_{12}, α_{23}, and α_{34} as in fig. 1. Then the other roots are all of the edges of the bigger

Date: October 25, 2018.

[1]Of course until the last one. . .

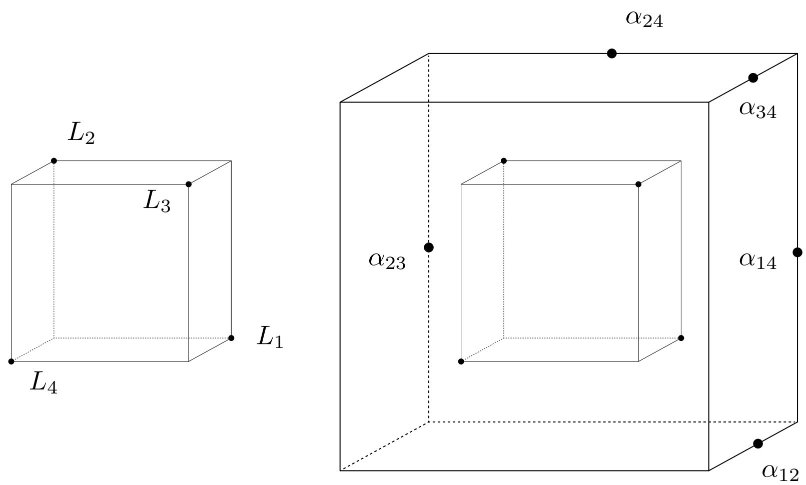


FIGURE 1. (Left) The weights of the standard representation of $\mathfrak{sl}(4,\mathbb{C})$. (Right) The weights of the adjoint representation of $\mathfrak{sl}(4,\mathbb{C})$.

cube on the right of fig. 1. The point here is that if we consider a matrix with a 1 in the spot:

$$\begin{pmatrix} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ & & \alpha_{23} & \alpha_{24} \\ & & & \alpha_{34} \\ & & & \end{pmatrix}$$

it will be an eigenvector with eigenvalue as in the right of fig. 1. Then there are six more on the other edges of the cube.

The cone of elements which pair positively with the positive roots can be seen in fig. 2. This is somehow a triangle on the back face coned off to the origin. Note that it takes 24 such triangles to cover the face of the cube, which is of course what we would expect since $|S_4| = 24$, which makes sense since these cones should correspond to the Borel subalgebras which are acted on simply freely by the Weyl group.

We can view this cone as coming from three copies of $\mathfrak{sl}(3,\mathbb{C})$ as being the sort of intersections of the three figure in fig. 3. Reflections over these planes are the simple reflections for $\mathfrak{sl}(4,\mathbb{C})$.

3. Representations of $\mathfrak{sp}(4,\mathbb{C})$

3.1. **Definition.** Recall the Lie algebra $\mathfrak{sp}(2n,\mathbb{C}) \subseteq \mathfrak{sl}(2n,\mathbb{C})$, is the Lie algebra of the Lie group $\mathrm{Sp}(2n,\mathbb{C}) \subseteq \mathrm{SL}(2n,\mathbb{C})$ which is the group of $2n \times 2n$ matrices which preserve the standard symplectic form in the sense that

$$\mathrm{Sp}(2n,\mathbb{C}) = \left\{ A \in \mathrm{SL}(2n,\mathbb{C}) \mid A^T J A = J \right\}$$

where we have fixed a symplectic form

$$\omega = \sum_i e_{2i-1} \wedge e_{2i}$$

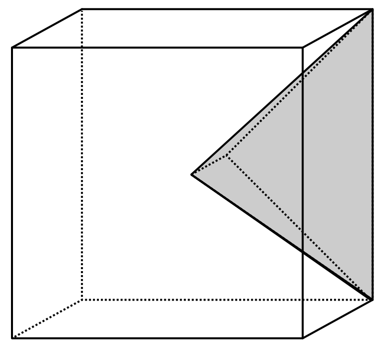


FIGURE 2. The cone of dominant weights in $\mathfrak{sl}(4,\mathbb{C})$.

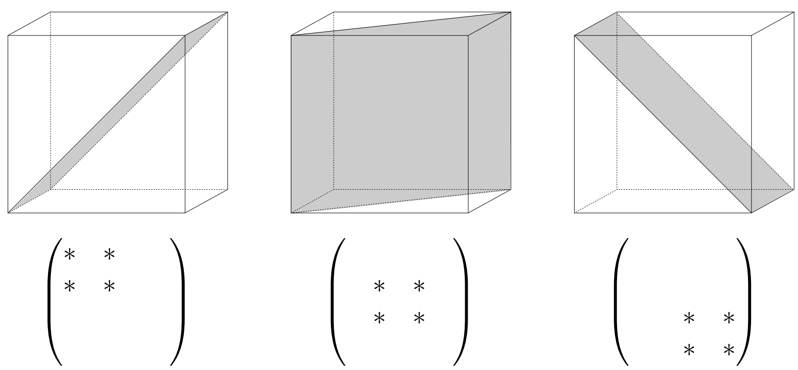


FIGURE 3. The three Levi $\mathfrak{sl}(2,\mathbb{C})$s living inside of $\mathfrak{sl}(4,\mathbb{C})$ give us reflections over these three planes.

which, as a matrix, looks like

$$J = \begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 0 & -1 \\ & & 1 & 0 \end{pmatrix}$$

But it doesn't really matter, as long as we take something skew-symmetric and non-degenerate, since we have the following.

Exercise 1. Show that all symplectic forms are equivalent.

Then the Lie algebra is

$$\mathfrak{sp}(2n,\mathbb{C}) = \left\{ X \in \mathfrak{sl}(n,\mathbb{C}) \mid JX = -X^T J \right\}$$

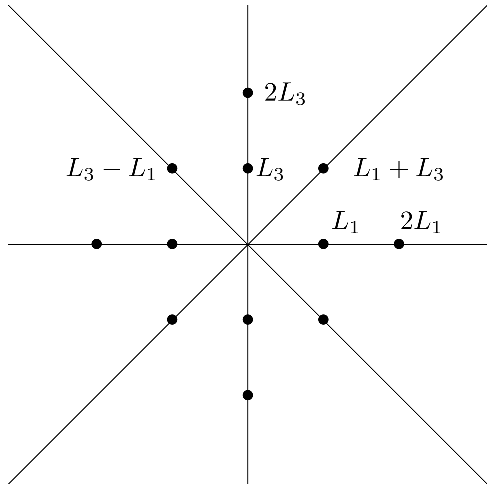


FIGURE 4. The dual space $\mathfrak{h}^*$ of our Cartan subalgebra with weights L_1, L_3, and the eight roots.

Exercise 2. Show that $\mathrm{Sp}(2n, \mathbb{C})$ is simply connected, so this is indeed the unique simply connected Lie group associated to this Lie algebra. Also show $Z(\mathrm{Sp}(2n, \mathbb{C})) = \mathbb{Z}/2$.

3.2. **Roots.** We can take our Cartan subgroup to be

$$H = \left\langle \begin{pmatrix} a & & & \\ & a^{-1} & & \\ & & b & \\ & & & b^{-1} \end{pmatrix} \right\rangle$$

The idea is that if we are going to preserve the area, we need to spin neighbouring coordinates by opposite amounts. Technically we should check that this is not only abelian but actually maximal abelian, but we know this is rank 2, and we've already seen the classification so we already know this is maximal.

This means our Cartan subalgebra $\mathfrak{h}$ is

$$\mathfrak{h} = \left\langle \begin{pmatrix} r & & & \\ & -r & & \\ & & s & \\ & & & -s \end{pmatrix} \right\rangle$$

Again we have $\mathfrak{h}^* \simeq \mathbb{C}^2$, so we have an analogous picture in fig. 4.

Life is a little better here than it was in $\mathfrak{sl}(3, \mathbb{C})$, because we have two favorite functionals. We can take the functional which returns out the first diagonal entry, L_1, and the functional which returns the third, L_3. Now we can generate the roots by calculating commutators, and draw them as in fig. 4. The Weyl group here is $S_2 \times \mathbb{Z}/2 = \mathbb{Z}/2 \times \mathbb{Z}/2$ which is of course the dihedral group D_2. Now we want to find positive and simple roots. To do this we pick the Borel:

$$B = \mathrm{Sp}(2n, \mathbb{C}) \cap \{ \text{ upper triangular matrices } \}$$

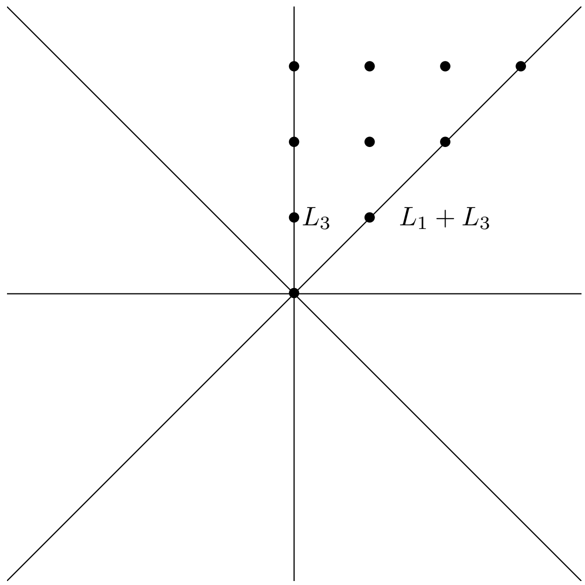


FIGURE 5. The dominant weight lattice of $\mathfrak{sp}(4, \mathbb{C})$.

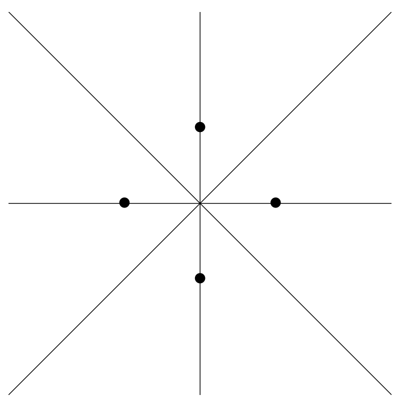


FIGURE 6. Weights of the standard representation of $\mathfrak{sp}(4, \mathbb{C})$.

Then the positive roots are $L_3 - L_1$, $2L_3$, $L_1 + L_3$, and $2L_1$, and out of these the two simple roots are $2L_1$ and $L_3 - L_1$. The dominant weights are then $\mathbb{Z}$ multiples of $L_1 + L_3$ and L_3 as in fig. 5.

3.3. **Constructing representations.** Now we want to construct some representations of this by hand. Recall in $\mathfrak{sl}(3, \mathbb{C})$, we constructed the standard representation and the dual standard representation, and then we could just get everything from tensoring these. Analogously, the two most important constructions here have highest-weights L_3 and $L_1 + L_3$. The representation corresponding to L_3 is the standard representation $\mathbb{C}^4$. The weights of this are as in fig. 6.

Now for $L_1 + L_3$, we should first notice that the weights should likely lie in some sort of convex hull that looks like a square, where we have just reflected this

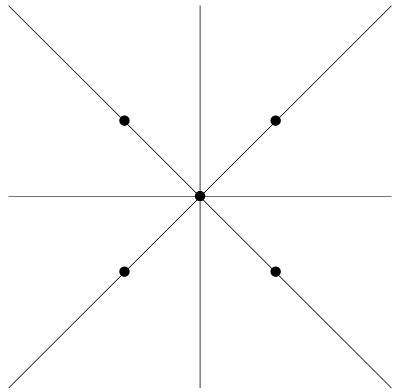
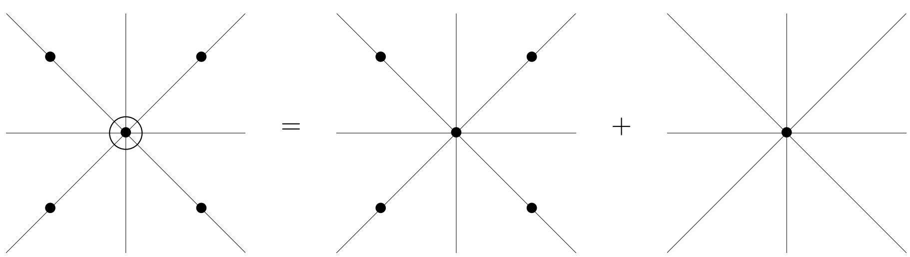


FIGURE 7. The weights of the representation with highest-weight $L_1 + L_3$.

highest weight across these hyperplanes. Then we might wonder if 0 is a weight of this representation. To find out, we can just act with the "lowering" operators and see if we land in it. Applying the root $-L_1 - L_3$, we do land at 0, so it is possible. The answer turns out to be as in fig. 7. So now we know the weights, and we want to find the actual representation. We learned from $\mathfrak{sl}(n, \mathbb{C})$, that once we know the standard representation, the smaller ones are just exterior powers. This inspires us to look at:

$$\wedge^2 \left(\mathbb{C}^4\right) = \mathbb{C} \cdot \omega \oplus W$$

which is a 6-dimensional representation, where W is some five-dimensional irreducible representation. The weights of this exterior power are pairwise sums of weights from the standard where we don't add any weight to itself, so we get a weight of multiplicity 2 at 0, and one at each of the four corners. Then the weights of the decompositions are as follows:



Note that $\mathfrak{sp}(4, \mathbb{C})$ is born as a subalgebra of $\mathfrak{sl}(4, \mathbb{C})$. This means we can project the dual space of the Cartan subalgebras

$$\begin{array}{c} \mathfrak{h}^*_{\mathfrak{sl}(4,\mathbb{C})} \\ \downarrow \\ \mathfrak{h}^*_{\mathfrak{sp}(4,\mathbb{C})} \end{array}$$

according to the dual of the inclusion $\mathfrak{sp} \hookrightarrow \mathfrak{sl}$. Then if we would have picked our basis correctly, the eigenvalues would map exactly to eigenvalues $L_i \mapsto L_i$.

Exercise 3. Draw G_2. This is the other distinct rank two simple Lie algebra.

4. Flag variety and fundamental affine space for $\mathfrak{sp}(4,\mathbb{C})$

In general, the flag variety X should be the moduli of Borel subalgebras $\mathfrak{b} \subset \mathfrak{g}$. For $\mathrm{SL}(n,\mathbb{C})$ we saw that $X \simeq G/B$ since $G \circlearrowright X$ transitively by conjugation with stabilizer B. In this setting we have the following:

Proposition 1. *The flag variety for* $\mathfrak{sp}(4,\mathbb{C})$ *is*

$$X \simeq \left\{\{0\} \subseteq E_1 \subseteq E_2 \subseteq \cdots \subseteq E_n \subseteq \mathbb{C}^{2n}\right\}$$

where E_i *is isotropic*[2]*, i.e.* $\omega|_{E_i} = 0$ *and* E_n *is Lagrangian.*

Exercise 4. Show that the symplectic group acts transitively on these isotropic flags, and that the stabilizer of the standard isotropic flag is exactly a Borel subgroup.

Example 1. For $\mathfrak{sp}(2,\mathbb{C})$, the flag variety is just $\mathbb{CP}^1$, which is good since $\mathfrak{sp}(2,\mathbb{C}) \simeq \mathfrak{sl}(2,\mathbb{C})$, so they should agree.

Example 2. For $\mathfrak{sp}(4)$, the choices of E_1 are just $\mathbb{CP}^3$, i.e. forgetting E_1 is a map $X \to \mathbb{CP}^3$. Then the fiber is $\mathbb{CP}^1$, which is the choice of E_2 for a fixed E_1. The idea is that once we fix E_1, we are looking for lines symplectically orthogonal to it. So they have to somehow live in $\mathfrak{sp}(2)/B \simeq \mathbb{CP}^1$.

[2]Note that all lines are isotropic, so it's really only a relevant condition for $i \geq 1$.

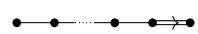
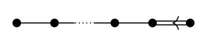
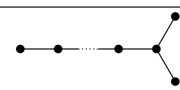
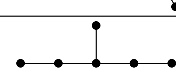
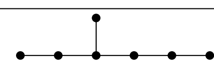
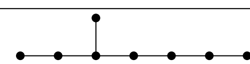
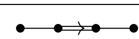
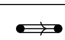
LECTURE 19

LECTURES BY: DAVID NADLER
NOTES BY: JACKSON VAN DYKE

Today we will talk a bit more about the classification of semisimple Lie algebras, root systems, and Dynkin diagrams. Going forward, we will take a more geometric approach via D-modules.

1. Classification of simple Lie algebras

Recall we saw the following classification of Lie algebras according to their rank.[1]

	$\mathfrak{g}$	Diagram	$Z(G)$	$\pi_1(G)$
$A_n\ (n \geq 1)$	$\mathfrak{sl}(n+1, \mathbb{C})$		$\mathbb{Z}/(n+1)\mathbb{Z}$	0
$B_n\ (n \geq 2)$	$\mathfrak{so}(2n+1, \mathbb{C})$		0	$\mathbb{Z}/2\mathbb{Z}$
$C_n\ (n \geq 3)$	$\mathfrak{sp}(2n, \mathbb{C})$		$\mathbb{Z}/2\mathbb{Z}$	0
$D_n\ (n \geq 4)$	$\mathfrak{so}(2n, \mathbb{C})$		$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
E_6	–		$\mathbb{Z}/3\mathbb{Z}$	–
E_7	–		$\mathbb{Z}/2\mathbb{Z}$	–
E_8	–		0	–
F_4	–		0	–
G_2	–		0	–

We will now see what these cartoons mean mathematically. The strategy will be to go from simple Lie algebras, extract root systems, and get a list of Lie algebras out of that.

1.1. Root systems.

Definition 1. A root system is a real euclidean[2] vector space $(V, \langle \cdot, \cdot \rangle)$ equipped with some subset of roots R which satisfy the following properties:

(1) The roots span V.
(2) If $\alpha \in R$ then $-\alpha \in R$.
(3) α and $-\alpha$ are the only roots on $\mathbb{R} \cdot \alpha$.

Date: November 6, 2018.

[1] Recall the rank of a Lie algebra is just the dimension of a Cartan subalgebra. These of course all have to be the same since they're related by conjugation by the unique connected, simply-connected Lie group.

[2] Non-degenerate inner product.

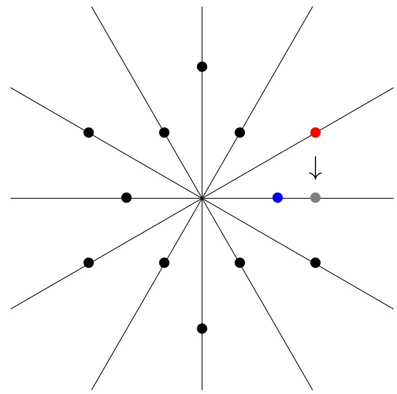


FIGURE 1. The root system G_2. Note that the projection of the red root onto the horizontal axis is 1.5 times the blue root. The projection is pictured in gray.

(4) Reflection across $\alpha^\perp$ for $\alpha \in R$ preserves the set of roots.
(5) Orthogonal projection to $\mathbb{R} \cdot \alpha$ takes R to $\{\pm\alpha, \pm\alpha/2, 3\alpha/2\}$.

Example 1. The roots of $\mathfrak{sl}(3, \mathbb{C})$ and $\mathfrak{sp}(4, \mathbb{C})$ comprise root systems.

Proposition 1. *If $\mathfrak{g}$ is a semisimple complex Lie algebra, and $\mathfrak{h} \subseteq \mathfrak{g}$ is a Cartan subalgebra, then the roots $R \subset \mathfrak{h}^*$ satisfy all the axioms of a Root system in $V = \mathrm{Span}_{\mathbb{R}}(R) \subseteq \mathfrak{h}^*$ with respect to the Killing inner product.*

Remark 1. If axiom 1 fails for the roots of some Lie algebra, then this means $\mathfrak{g}$ must have a nontrivial center, so it can't be semisimple. Motivation from the second axiom is meant to come from the fact that for every root which somehow comes from an upper triangular location, there should be a corresponding root which came from a lower triangular location. This is again because we are semisimple here. If we were talking about solvable Lie algebras, we would somehow only have roots on one side. For axiom 4, recall we have the Weyl group action. For any root, we have a copy of $\mathfrak{sl}(2, \mathbb{C})$ with that root as its root. Then we can reflect according to the this particular $\mathfrak{sl}(2, \mathbb{C})$. To see the last axiom 5, consider any root γ, and reflect it with respect to α. Then we need the difference between γ and its reflections to be in the $\mathbb{Z}$-linear span of the roots.

Example 2. So far we have only ever seen root systems where we only need $\pm\alpha$ and $\pm\alpha/2$ in the list in axiom 5. An example that illustrates the fact that we need the $3\alpha/2$ which can be seen in fig. 1.

As it turns out, the arrow between semi-simple Lie algebras and root systems is really an equivalence.

Lemma 1. *There is a map from based root systems*[3] *to root systems. I.e. we can construct everything from the simple roots.*

[3] These just consist of a basis of roots satisfying some properties.

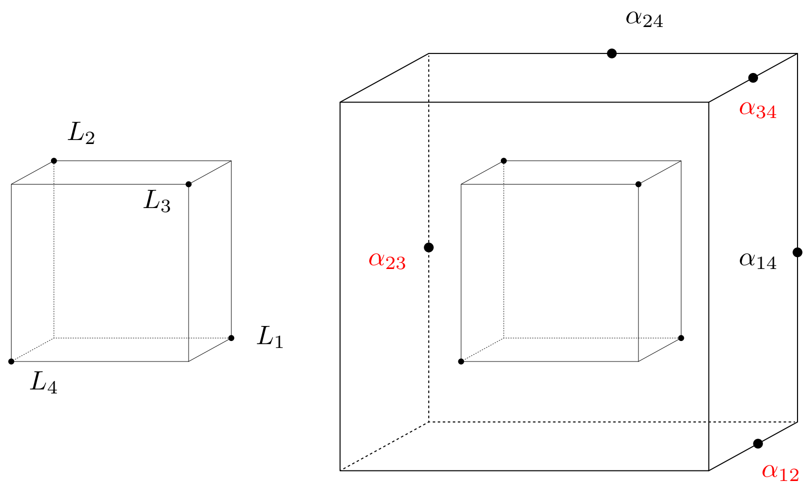


FIGURE 2. (Left) The weights of the standard representation of $\mathfrak{sl}(4,\mathbb{C})$. (Right) The roots of $\mathfrak{sl}(4,\mathbb{C})$, i.e. the weights of the adjoint representation of $\mathfrak{sl}(4,\mathbb{C})$. One choice of simple roots is in red.

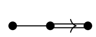
1.2. **Dynkin diagrams.** A Dynkin diagram is a graph with a vertex for every $\alpha \in \Delta$. Then there is a single edge edge for an angle between the roots of $2\pi/3$, a double edge for $3\pi/4$, and a triple edge for $5\pi/6$. The direction of the double and triple edges point from longer to shorter roots.

Example 3. Recall $\mathfrak{sl}(4,\mathbb{C})$ has a picture as in fig. 2. One choice of simple roots consists of α_{12}, α_{23}, and α_{34}. The angle between α_{12} and α_{34} is $\pi/2$ so they don't get connected, but the angle between α_{12} and α_{23} is $2\pi/3$, and similarly for α_{23} and α_{34}, so we indeed get

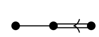
which is the A_3 diagram.

Example 4. First consider $\mathfrak{sp}(6,\mathbb{C})$ which has root system B_3 as in fig. 3. $\mathfrak{so}(7,\mathbb{C})$ has the root system C_3 as in fig. 3. Both of these systems have roots on all edges of the square. The B_3 root system has roots on all surfaces of the cube, whereas the C_3 system has roots above all surfaces of the cube. Therefore they have the same angular relationship, which is that two of the simple roots have an angle of $\pi/2$, two have an angle $2\pi/3$, and two have an angle of $3\pi/4$. But the relationship between the lengths of the roots is different which is why the direction of the arrow is different for B_3 versus C_3:

B_3



C_3



Example 5. For $\mathfrak{so}(4,\mathbb{C})$ the roots are as in fig. 4. which means the angle between all roots is $\pi/2$, so the Dynkin diagram is just two unconnected points. This is a reflection of the fact that

$$\mathfrak{so}(4,\mathbb{C}) = \mathfrak{sl}(2,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C})$$

Exercise 1. Come up with a creative way to think about/draw F_4.

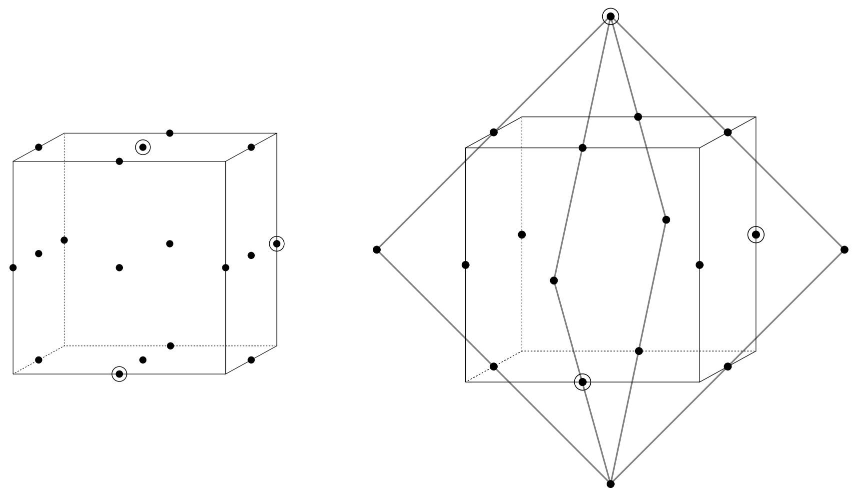


FIGURE 3. (Left) The root system B_3 for $\mathfrak{sp}(6,\mathbb{C})$. The simple roots are the roots of multiplicity 2. (Right) The root system C_3 for $\mathfrak{so}(7,\mathbb{C})$. The simple roots are the roots of multiplicity 2.

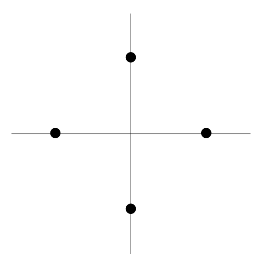


FIGURE 4. The roots of $\mathfrak{so}(4,\mathbb{C})$.

1.3. **Simply-laced Dynkin diagrams.** The A_n, D_n, and E_n Dynkin diagrams are called simply-laced because they only have single bars in their Dynkin diagram. One sample connection to another part of mathematics is as follows.[4] The ADE diagrams are in bijection with rational/du Val surface singularities.

Example 6. The diagrams A_n correspond to

$$V\left(x^2+y^2+z^{1+n}\right)\subseteq\mathbb{C}^3$$

For $n=1$, so $\mathfrak{sl}(2,\mathbb{C})$, we get a nice cone. As n increases this gets worse and worse. Now we might ask how far this is from being a manifold. One way to measure this is to find a minimal resolution $\tilde{X}_n\to X_n$. The idea is that $\tilde{X}_n$ will be a smooth surface, and this map will be an isomorphism away from the singular point which is also proper everywhere.[5] Then the preimage of the singular point is n copies of

[4] This is typically attributed to to Grothendieck but Professor Nadler says the way it goes is that the rich get richer.

[5] This just means the inverse image of compact sets is compact.

$\mathbb{CP}^1$ which intersect in a chain which is the A_n diagram. The same story holds for types D_n and E_n, only in those cases the singularities look a bit different.

2. Harish Chandra center

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$, and let G be the connected simply-connected group with Lie algebra $\mathfrak{g}$. Since $\mathfrak{g}$ is simple, the center $\mathfrak{z}(\mathfrak{g}) = \langle 0 \rangle$. This means $Z(G)$ must be finite.[6] This seems like we can't attack it with much since it's sort of atomic, but as it turns out the enveloping algebra $\mathcal{U}\mathfrak{g}$ has a somehow large center. This is an amazing fact, because if we have a G action, we have a $\mathfrak{g}$ action, so we have a $\mathcal{U}\mathfrak{g}$ action, and this object actually has this large useful center.

Theorem 1 (Harish Chandra)**.** *The center of the enveloping algebra $\mathfrak{z}(\mathcal{U}\mathfrak{g})$ is isomorphic to $(\mathcal{U}\mathfrak{g})^{(W,\tilde{\cdot})}$ where W acts around ρ rather than the origin.*

[6] It is immediate that the center must be discrete, but after a bit of work we could see that it does indeed have to be finite.

LECTURE 20
MATH 261A

LECTURES BY: PROFESSOR DAVID NADLER
NOTES BY: JACKSON VAN DYKE

1. Harish Chandra center

We will continue our discussion of the Harish-Chandra center. The setup will be $\mathfrak{g}$ a semi-simple complex Lie algebra, and G the unique connected, simply-connected Lie group. Inside $\mathfrak{g}$ we have a Cartan subalgebra $\mathfrak{h}$, and on this we have this Weyl group action. Then we have the following theorem:

Theorem 1.
$$\mathfrak{z}\left(\mathcal{U}\mathfrak{g}\right) \simeq \left(\mathcal{U}\mathfrak{h}\right)^{W,\tilde{\cdot}}$$

2. The RHS

The first thing to notice is that we have a canonical isomorphism $\mathcal{U}\mathfrak{h} \simeq \mathrm{Sym}\left(\mathfrak{h}\right)$. Recall
$$\mathcal{U}\mathfrak{h} \coloneqq \bigoplus_{n\geq 0} \mathfrak{h}^{\otimes n} / \left(H_1 \otimes H_2 - H_2 \otimes H_1 - [H_1, H_2] = 0\right).$$
Since $\mathfrak{h}$ is abelian, we are actually just killing transpositions, so we get a symmetric algebra.

2.1. **A useful point of view.** A useful point of view is to think of $\mathrm{Sym}\,\mathfrak{h}$ as polynomial functions on $\mathfrak{h}^*$, written $\mathbb{C}\left[\mathfrak{h}^*\right]$. Now we have a W action[1] only we want to re-center at $-\rho$ to get the this $\tilde{\cdot}$ action.

To get this action we had to choose a Borel subalgebra, which told us the positive roots, and then we write down half the sum of the positive roots, and this is ρ. More specifically,
$$w \tilde{\cdot} \lambda = w \cdot \left(\lambda + \rho\right) - \rho \ .$$
The reason this point of view is useful, is that we have
$$\left(\mathcal{U}\mathfrak{h}\right)^{W,\tilde{\cdot}} = \mathrm{Sym}\left(\mathfrak{h}\right)^{W,\tilde{\cdot}} = \mathbb{C}\left[\mathfrak{h}^*\right]^{W,\tilde{\cdot}}$$
so we should think of this as consisting of the polynomials invariant under this twisted W action.

Remark 1. This $\mathbb{C}\left[\mathfrak{h}^*\right]^{W,\tilde{\cdot}}$ is what we would write in algebraic geometry as
$$\mathbb{C}\left[\mathfrak{h}^* // W, \tilde{\cdot}\right]$$
Note that this space, which we now write as
$$\mathfrak{c}^* \coloneqq \mathfrak{h}^* // W \ ,$$

Date: November 8, 2018.

[1] Professor Nadler's polarity changed after his week away and now he gets shocked by the blackboard.

is again an affine space.

Fact 1 (Fantastic fact)**.** $(\mathcal{U}\mathfrak{h})^{W,\tilde{\cdot}}$ *is again a polynomial algebra.*

Example 1. In $\mathfrak{sl}(n,\mathbb{C})$, $W = \Sigma_n$ is the symmetric group. We will ignore the $\tilde{\cdot}$ part for now just to get a feeling for this. Here $\mathfrak{h}^*$ just consists of n-tuples of eigenvalues of trace 0. Now we can ask for all of the polynomial functions of the eigenvalues that are invariant under their permutations. As it turns out, these are the polynomials in the elementary symmetric functions:

$$\mathbb{C}[\mathfrak{h}^*//W] \simeq \mathbb{C}[\sigma_2, \cdots, \sigma_n] \ .$$

So we should think of this guy as another affine space.

2.2. **Motivation/what this is telling us.** Let's stop for a second and enjoy what this theorem is telling us. It is saying that any time you write down a module over this algebra $\mathcal{U}\mathfrak{g}$, i.e. a representation of $\mathfrak{g}$, finite or infinite dimensional, you have an action of the center on the module. So the entire representation theory of this algebra lives over functions on an affine space. So we can talk about when a module is supported at a point of an affine space, based on acting on it by functions.

The upshot of all this is that the category $\mathcal{U}\mathfrak{g}$-**Mod** is linear over the polynomial algebra $\mathbb{C}[\mathfrak{c}^*]$. So if we are given some $\mathcal{U}\mathfrak{g}$ module M, we can take some[2] $\bar{\lambda} \in \mathfrak{c}^*$, and then consider functions which vanish exactly at this point, and multiply the module by them. Now we can ask if the action is, say, always an isomorphism. And if it is, that means the module would somehow live only here.

Remark 2. What professor Nadler is trying to convey here in basic terms is the following. If you've ever taken Spec of a module in algebraic geometry you know you get a sheaf on this thing. So it tells us that all modules have some expansion over this affine space.

In particular, if you're irreducible, we already know what the irreducible modules are for a polynomial algebra. They're just given by the maximal ideals, i.e. just the points. So the upshot is that all of the irreducible modules live over points of $\mathfrak{c}^*$. If a module "spreads out" then we can just multiply it by a function $x - \bar{\lambda}$ and get a submodule.

So again, the representation theory of this lives over an affine space, whose points are somehow sets of eigenvalues. So what remains for us, as representation theorists trying to understand all $\mathcal{U}\mathfrak{g}$ modules, is to fix any single point in $\mathfrak{c}^*$ and just study all of the modules that live above this.

For fixed $\bar{\lambda} \in \mathfrak{c}^*$ we set the following notation:

$$\mathcal{U}_{\bar{\lambda}}\mathfrak{g} = \mathcal{U}\mathfrak{g}/I_{\bar{\lambda}}$$

where $I_{\bar{\lambda}}$ is the ideal of $\mathbb{C}[\mathfrak{c}^*]$ which consists of functions vanishing at $\bar{\lambda}$. So the only nonzero things are what's happening here.

Example 2. This is supposed to be like taking $k_\lambda \simeq k[t]/(t-\lambda)$ to get the sky scraper (copy of scalars) at λ.

So now we have the following observation: $\mathcal{U}_{\bar{\lambda}}\mathfrak{g}$-**Mod** just consists of $\mathcal{U}\mathfrak{g}$-**Mod** on which the center acts by the character. I.e. we map

$$\mathfrak{z} \to \mathfrak{z}/I_{\bar{\lambda}} \simeq k_{\bar{\lambda}}$$

[2] We call this $\bar{\lambda}$ to convey that it somehow comes as the image of some set of eigenvalues. Points of $\mathfrak{h}^*$ are like ordered eigenvalues, and points of $\mathfrak{c}^*$ are like unordered eigenvalues.

Remark 3. What do we mean by acts by the character? If we take any $\mathcal{U}\mathfrak{g}$ module we can ask how the center acts. Remember we want to think of this as functions on $\mathfrak{c}^*$. So now we're just doing algebraic geometry. We have a polynomial algebra, and we're asking how it acts on a module. One of our favorite ways is to take a polynomial, restrict it to $\bar{\lambda}$ to get its value, and then scale the module by that value. But this is mathematically the same as saying the polynomial algebra maps to its quotient by the ideal of functions vanishing at $\bar{\lambda}$ which is just the values at $\bar{\lambda}$.

The whole representation theory of the enveloping algebra, which is to say the representation theory of $\mathfrak{g}$, has this giant center in it, and we can somehow talk about those representations that live over any point in Spec of the center. We define this algebra to be the quotient of the entire algebra which is given by just looking at the point $\bar{\lambda}$. The whole representation theory is then somehow an "integral" over the representation theory at these $\bar{\lambda}$.

Remark 4. This is somehow the reason one likes centers. If $\mathfrak{g}$ was commutative, it would just be its own center, and we would be back in algebraic geometry where we know how to classify irreducibles (given by maximal ideals) so there's a whole structure there. And the next best thing is the (very big) center of $\mathcal{U}\mathfrak{g}$. So in this part things are just algebraic geometry, and then for each $\bar{\lambda}$ you have to do the algebraic geometry of this new particular algebra which has center just consisting of scalars.

Remark 5. This is somehow a general paradigm. Any time someone gives you a mathematical object, you should ask what its endomorphisms are, and then find the center of the endomorphisms. Then spread it out over Spec of the center. Professor Nadler says you can understand almost everything in mathematics by asking that question.

2.3. Calculation for $\mathfrak{sl}(2,\mathbb{C})$.

Example 3. Let $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$. Then $\mathfrak{h} = \mathbb{C}\cdot H$ and $\mathfrak{h}^* = \mathbb{C}\cdot L_1$ where $L_a(aH) = a$. We want to think about L_1 as a point, and then functions on this line $\mathfrak{h}^*$ are polynomials in H, $\mathbb{C}[\mathfrak{h}^*] \simeq \mathbb{C}[H]$. Recall the usual $W = \mathbb{Z}/2 = \{1,\sigma\}$ action is just reflection wrt 0, so $\mathbb{C}[H]^W$ is all polynomials invariant under $H \mapsto -H$, which is of course $\mathbb{C}\left[H^2\right]$. This is again a polynomial algebra, and we should think of this as being like a double cover by the square map. Here $\rho = L_1$, so $-\rho = -L_1$, so the action of $W = \mathbb{Z}/2$ by $\tilde{\cdot}$ is really reflection over -1:

$$\sigma\tilde{\cdot}(aL_1) = -(aL_1 + L_1) - L_1 = (-a-2)L_1\ .$$

Now the invariant functions under this action are:

$$\mathbb{C}[\mathfrak{h}^*]^{W,\tilde{\cdot}} = \mathbb{C}\left[(H+1)^2\right]$$

Remark 6. One might be annoyed by this $\tilde{\cdot}$ action because it's an extra thing to keep track of. Professor Nadler says that often times in mathematics it is best to respect structures like this and be their friends so they can guide us.

3. The LHS

Now we want to think about the LHS of the theorem. Recall again that:

$$\mathcal{U}\mathfrak{g} = \left(\bigoplus_{n=0}^{\infty} \mathfrak{g}^{\otimes n}\right) / \left(X\otimes Y - Y\otimes X - [X,Y]\right)$$

But this quotient doesn't respect the grading of the tensor algebra since it effectively sets degree 2 things equal to degree 1 things. We do however maintain the filtration:

$$F^0\mathcal{U}\mathfrak{g} \subset F^1\mathcal{U}\mathfrak{g} \subset \cdots$$

where $F^i\mathcal{U}\mathfrak{g}$ somehow consists of tensors of degree i and lower. We can canonically write that

$$F^0\mathcal{U}\mathfrak{g} = \mathbb{C} \qquad\qquad F^1\mathcal{U}\mathfrak{g} = \mathbb{C} \oplus \mathfrak{g}$$

are tensors of degree 0 and degree 1, but after this there is no canonical splitting of the filtration. PBW does however give us a non-canonical splitting.

Recall that this PBW story was that $\mathcal{U}\mathfrak{g}$ has a basis of ordered monomials of the form $X^{\alpha_1}H^{\alpha_2}Y^{\alpha_3}$ which gives us a splitting of this filtration. This tells us that we can set

$$F^2\mathcal{U}\mathfrak{g} = \mathbb{C} \oplus \mathfrak{g} \oplus \{\text{degree two s.t. linear combination of PBW basis}\} \ .$$

Now if we pass to the associated graded algebra:

$$\mathrm{Gr}_F(\mathcal{U}\mathfrak{g}) = \bigoplus_{n=0}^{\infty} F^n\mathcal{U}\mathfrak{g}/F^{n-1}\mathcal{U}\mathfrak{g} \simeq \mathrm{Sym}(\mathfrak{g})$$

we get a symmetric algebra on $\mathfrak{g}$.

3.1. **Organizing $\mathcal{U}\mathfrak{g}$ and $\mathrm{Sym}(\mathfrak{g})$.** What we did above doesn't really have anything to do with PBW. Any time we have a filtered algebra like this we can do what's called the Rees construction, which builds a new algebra which depends on both the initial algebra and the filtration. Explicitly we define the algebra:

$$R\mathfrak{g} = \bigoplus_{n=0}^{\infty} F^n\mathcal{U}\mathfrak{g} \cdot \hbar^n$$

where the multiplication is given by:

$$(\tau_n \cdot \hbar^n) \cdot (\tau'_m \cdot \hbar^m) := \tau_n\tau'_m \cdot \hbar^{n+m}$$

Note that $\hbar$ is central, which means this algebra lives over the $\hbar$ line. Now we can ask about the fibers of this algebra at different points. When $\hbar = 0$, we recover the associated graded algebra $\mathrm{Gr}_F(\mathcal{U}\mathfrak{g})$, and at $\hbar = 1$ we recover $\mathcal{U}\mathfrak{g}$. In general we get:

$$R_{\mathfrak{g}}|_a = R\mathfrak{g}/(\hbar - a)$$

Exercise 1. Show that with this definition $R_{\mathfrak{g}}|_0 = \mathrm{Gr}_F(\mathcal{U}\mathfrak{g})$ and $R_{\mathfrak{g}}|_1 = \mathcal{U}\mathfrak{g}$.

Solution. For $\hbar = 0$ we can look at the inclusion of something in the ideal $(\mathfrak{h})$ into $F^{n+1}\mathcal{U}\mathfrak{g} \cdot \hbar$ and this will be exactly $\mathrm{Gr}_F(\mathcal{U}\mathfrak{g})$.

Remark 7. The whole point here is that what we did above is completely general and doesn't have anything to do with PBW.

Remark 8. So we start with a symmetric algebra, the functions on $\mathfrak{g}$, and then one can view this as quantizing those functions. A useful point of view is that $R\mathfrak{g}$ is the deformation quantization[3] of the algebra of symmetric functions with respect to the Poisson structure given by $[\cdot,\cdot]$.

[3] This is just a word for deforming a commutative algebra to be something noncommutative.

Remark 9. The general point of view here is that when we're studying $\mathcal{U}\mathfrak{g}$, we're really looking at $\mathfrak{g}^*$ as a vector space, which has a ring of functions with a Poisson bracket, and we're quantizing the Poisson bracket to get $\mathcal{U}\mathfrak{g}$.

3.2. **Key observation.** Now we want to determine the center of $\mathcal{U}\mathfrak{g}$. What is the center of the special fiber? This is commutative, so it is the whole thing. So passing between them something strange happens since the center gets much smaller. We now have the following key observation:

$$\mathfrak{z}(\mathcal{U}\mathfrak{g}) \simeq (\mathcal{U}\mathfrak{g})^G$$

under the adjoint action. So if we take a tensor τ such that $\operatorname{Ad}_g \tau = \tau$, then differentiating with respect to $g = \exp(tX)$ we get $\operatorname{ad}_X(\tau) = 0$ which means $[X, \tau] = 0$ where this extends by the Jacobi identity. Therefore, being an invariant means you're certainly in the center. Now conversely, if something is in the center we can just exponentiate it, which generates a neighborhood of the identity, and therefore this thing is invariant under the adjoint action of G as well. So we just reinterpreted being in the center as a quality which only concerns invariants of this vector space.

Now the PBW splitting gives the isomorphism of vector spaces $\mathcal{U}\mathfrak{g} \simeq \operatorname{Sym}(\mathfrak{g})$ so the subspaces of invariants are the same as vector spaces:

$$(\mathcal{U}\mathfrak{g})^G \simeq \operatorname{Sym}(\mathfrak{g})^G \quad .$$

Warning 1. The LHS is commutative and the RHS is not, so these are certainly not isomorphic as algebras, and in fact they're not even isomorphic as G-representations. We will see an example of this soon.

Now there is a theorem of Chevalley[4] which says:

$$(\operatorname{Sym}\mathfrak{g})^G \simeq (\operatorname{Sym}\mathfrak{h})^W$$

Example 4. In $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, this isn't so surprising. In this case $\operatorname{Sym}(\mathfrak{g})^G$ consists of all polynomial functions on traceless $n \times n$ matrices which are invariant under change of basis, i.e. they are conjugation invariant. On the other hand, $\operatorname{Sym}(\mathfrak{h})^W$ consists of symmetric functions on traceless n-tuples of eigenvalues. There is a natural map $\operatorname{Sym}(\mathfrak{g})^G \to \operatorname{Sym}(\mathfrak{h})^W$ where we just restrict to the diagonal matrices. In the other direction, one needs to convince oneself that any function of $n \times n$ matrices that are conjugation invariant is just going to be a symmetric function on the eigenvalues.

Exercise 2. Show this.

All together we have shown that:

$$\mathfrak{z}(\mathcal{U}\mathfrak{g}) \simeq (\operatorname{Sym}\mathfrak{h})^W$$

as vector spaces, which shows us they are somehow the same size, which is what Harish-Chandra said was true. But there is no $\tilde{\cdot}$ in sight, so we need to go back and somehow correct the fact that this was not an isomorphism of algebras, or even of representations. We will see what this looks like for $\mathfrak{sl}(2, \mathbb{C})$, and the main point will be that we need to go and symmetrize the PBW basis. This changes the W action, which finishes the picture. In the process we will calculate the first casimir, which is to say the first interesting invariant.

[4] This is often referred to as Chevalley's restriction theorem.

LECTURE 21
MATH 261A

LECTURES BY: PROFESSOR DAVID NADLER
NOTES BY: JACKSON VAN DYKE

1. Harish Chandra center

Recall we are exploring the Harish Chandra center, and the isomorphism given by the theorem:

Theorem 1. *For $\mathfrak{g}$ a reductive complex Lie algebra,*

$$\mathfrak{z}\left(\mathcal{U}\mathfrak{g}\right) \simeq \mathbb{C}\left[\mathfrak{h}^*\right]^{(W,\tilde{\cdot})}$$

Recall that one way to think of the RHS is as $\mathbb{C}\left[\mathfrak{h}^* / \left(W, \tilde{\cdot}\right)\right]$. And then one of the main results of geometric representation theory is Chevalley's theorem which tells us that $\mathbb{C}\left[\mathfrak{h}^*\right]^{(W,\tilde{\cdot})}$ is a polynomial algebra which implies this quotient $\mathfrak{c}^*$ is an affine space. The point is, the center is functions on unordered eigenvalues.

Remark 1. The LHS is important because any time we want to study modules over something, we can ask what its center is and how it acts. The Algebraic geometers will succinctly say that $\operatorname{Spec}\left(\mathfrak{z}\left(\mathcal{U}\mathfrak{g}\right)\right) \simeq \mathfrak{h}^* / \left(W, \tilde{\cdot}\right)$.

Recall the key idea is that we have the following isomorphism as vector spaces:

$$\mathfrak{z}\left(\mathcal{U}\mathfrak{g}\right) \simeq \left(\mathcal{U}\mathfrak{g}\right)^G \quad .$$

This was because if we write

$$\mathrm{Ad}_g\left(v_1 \otimes \cdots v_k\right) = \mathrm{Ad}_g\left(v_1\right) \otimes \cdots \otimes \mathrm{Ad}_g\left(v_k\right)$$

then differentiating this gives us that the ad action is trivial on such elements. Then using PBW we saw $\left(\mathcal{U}\mathfrak{g}\right) \simeq \left(\operatorname{Sym}\mathfrak{g}\right)$ as vector spaces which means we have

$$\left(\mathcal{U}\mathfrak{g}\right)^G \simeq \left(\operatorname{Sym}\mathfrak{g}\right)^G$$

as vector spaces. Then Chevalley tells us that this is a polynomial algebra, and in fact:

$$\left(\mathcal{U}\mathfrak{g}\right)^G \simeq \operatorname{Sym}\left(\mathfrak{h}\right)^W$$

so they have sort of the same size.

Example 1. We will see why this is true for $\mathfrak{sl}(2,\mathbb{C})$. As usual $\mathfrak{sl}(2,\mathbb{C}) = \mathbb{C}\langle X, H, Y\rangle$. First we look for invariants in $\operatorname{Sym}\left(\mathfrak{g}\right)^G$. We know $\mathbb{C}$ is an invariant, and the adjoint representation $\mathfrak{g}$ is irreducible. Next we take the tensor:

$$\mathfrak{g} \otimes \mathfrak{g} = V_4 \oplus V_2 \oplus V_0 \simeq V_4 \oplus \mathfrak{g} \oplus \mathbb{C}$$

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and the interesting thing is that this $\mathbb{C}$ is a new invariant. Inside of $\mathfrak{g}^{\otimes 2} \simeq \operatorname{Sym}^2(\mathfrak{g}) \oplus \wedge^2 \mathfrak{g}$, the first thing to notice is that

$$\mathfrak{g}^{\otimes 2} \simeq \operatorname{Sym}^2(\mathfrak{g}) \oplus \wedge^2 \mathfrak{g} \simeq (V_0 \oplus V_4) \oplus V_2$$

so this $\mathbb{C}$ actually lives in $\operatorname{Sym}^2 \mathfrak{g}$. Now we have a favorite element of $\operatorname{Sym}^2 \mathfrak{g} \simeq \operatorname{Sym}^2 \mathfrak{g}^*$, which is the Killing form.

Exercise 1. Show that $V_0 = \mathbb{C}\langle \kappa \rangle$ where κ is the Killing form.

Solution. We can explicitly write the Killing form as $\kappa = H^2 + 4XY$. Now we want to see if this is invariant. It is in the zero weight space so we just need to calculate:

$$\begin{aligned}
[X, \kappa] &= \left[X, H^2\right] + [X, 4XY] \\
&= [X,H] \otimes H + H \otimes [X,H] + \frac{4}{2}\left(X \otimes [X,Y] + [X,Y] \otimes X\right) \\
&= -2X \otimes H - 2H \otimes X + 2H \otimes X + 2H \otimes X \\
&= 0
\end{aligned}$$

This is the unique invariant vector in $\operatorname{Sym}^2(\mathfrak{g})$.

We might keep searching for invariants, but Chevalley tells us that this is a polynomial algebra, so

$$\operatorname{Sym}(\mathfrak{g})^G = \mathbb{C} \oplus \mathbb{C} \cdot \kappa \oplus \mathbb{C} \cdot \kappa^2 \oplus \cdots \simeq \mathbb{C}[\kappa]$$

Now we need to lift this element into the enveloping algebra. So we seek a central element $\tilde{\kappa} \in \mathcal{U}\mathfrak{g}$ such that under the PBW isomorphism:

$$\mathfrak{z}(\mathcal{U}\mathfrak{g}) \simeq (\mathcal{U}\mathfrak{g})^G \simeq \mathbb{C}[\kappa]$$

we have that $\tilde{\kappa} \mapsto \kappa$. But now the ambiguity here is that we don't know where XY lifts, i.e. this depends on our choices in the PBW setup. But we can just choose $\tilde{\kappa} = H^2 + 2(XY + YX)$, and we already checked this is a good lift.

2. The ρ shift

Now let's find the ρ shift in this picture. Our goal is a more explicit isomorphism in the example of $\mathfrak{sl}(2, \mathbb{C})$. As usual, consider $G \circlearrowright G/N$ where N is the fundamental affine space. From this we get a map $\mathfrak{g} \to \mathbf{Vect}(G/N)$, and now we naturally get a map $\mathcal{U}\mathfrak{g} \to \operatorname{Diff}(G/N)$ where $\operatorname{Diff}(G/N)$ denotes the differential operators on G/N. We also have a commuting right H-action on G/N. All together, we have G/N with a left G action, and a right H-action, and these actions commute. So we get a map $\mathcal{U}\mathfrak{g} \otimes \mathcal{U}\mathfrak{h} \to \operatorname{Diff}(G/N)$.

Exercise 2. Prove that H is exactly the symmetries of G/N that commute with G.

Now if we take the center $\mathfrak{z}\mathfrak{g} \subseteq \mathcal{U}\mathfrak{g}$, then we have the following diagram:

$$\begin{tikzcd}
\mathfrak{z}\mathfrak{g} \arrow[d, hook] \arrow[dr] \arrow[drr, hook, "\varphi"] & & \\
\mathcal{U}\mathfrak{g} \arrow[r] & \operatorname{Diff}(G/N) & \mathcal{U}\mathfrak{h} \arrow[l, hook']
\end{tikzcd}$$

This factorization φ results from the facts:

(1) $\mathfrak{z}\mathfrak{g}$ commutes with $\mathcal{U}\mathfrak{g}$
(2) $\mathcal{U}\mathfrak{h}$ is exactly what commutes with $\mathcal{U}\mathfrak{g}$.

Then the Harish Chandra isomorphism is the map

$$\varphi : \mathfrak{z}\mathfrak{g} \to (\mathcal{U}\mathfrak{h})^{W,\tilde{}}$$

Example 2. We will do this explicitly for $\mathfrak{sl}(2, \mathbb{C})$. In this case $G/N \simeq \mathbb{C}^2 \setminus \{0\}$ with coordinates u and v, and with the left action, we get

$$H \mapsto -u\partial_u + v\partial_v \qquad X \mapsto -v\partial_u \qquad Y \mapsto -u\partial_v$$

Now let's calculate this Casimir $K = H^2 + 2(XY + YX)$:

$$\begin{aligned} K &= (u\partial_u - v\partial_v)^2 + 2(v\partial_u u\partial_v + u\partial_v v\partial_u) \\ &= (u\partial_u)^2 + (v\partial_v)^2 - u\partial_u v\partial_v - v\partial_v u\partial_u + 2(v\partial_u u\partial_v + u\partial_v v\partial_u) \\ &= (u\partial_u)^2 + (v\partial_v)^2 - uv\partial_u\partial_v - vu\partial_v\partial_u + 2(vu\partial_u\partial_v + v\partial_v + uv\partial_v\partial_u + u\partial_u) \\ &= (u\partial_u)^2 + (v\partial_v)^2 + 2uv\partial_u\partial_v + 2(u\partial_u + v\partial_v) \end{aligned}$$

Now for the right action,

$$H \mapsto u\partial_u + v\partial_v$$

and we want to write K as a polynomial in H, and indeed:

$$K = H^2 + 2H = (H+1)^2 - 1$$

which is of course ρ-shifted.

LECTURE 22
MATH 261A

LECTURES BY: PROFESSOR DAVID NADLER
NOTES BY: JACKSON VAN DYKE

1. $\mathfrak{sl}(3, \mathbb{C})$

Recall for $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})$ we have the Cartan subalgebra $\mathfrak{h} = \mathbb{C}\langle H_{12}, H_{23}, H_{31}\rangle$ where $H_{12} + H_{23} + H_{13} = 0$. Then $\mathfrak{h}^* = \langle L_1, L_2, L_3\rangle$, and we want to describe invariant functions on $\mathfrak{h}^*$. We can regard the H_{ij} as linear functions on $\mathfrak{h}^*$. For example, H_{12} is the function which is $+1$ at L_1, -1 at L_2, and zero on the span of L_3. Now to find functions invariant under the Weyl group action we want a basis of $\mathfrak{h}^*$ for which the Weyl group permutes the elements. Unfortunately the usual basis is not such a basis since, for example, (12) takes H_{12} to $-H_{12}$.

We need to take different coordinates in order for this to be a permutation action. The idea is that we don't want functions which are vanishing on the these particular hyperplanes. Instead we will set:

$$a = H_{12} - H_{31} \qquad b = H_{23} - H_{12} \qquad c = H_{31} - H_{23}\ .$$

Now these functions take values as in fig. 1.

Claim 1. $W = \Sigma_2$ permutes the functions a, b, and c.

"Proof" by example. Take $\sigma = (12)$, then this takes $a \mapsto b$, $b \mapsto a$, and $c \mapsto c$. $\square$

As a result of this observation, we can write:

$$\mathrm{Sym}(\mathfrak{h})^W \simeq \mathbb{C}[a, b, c]^W \simeq \mathbb{C}[\sigma_2, \sigma_3]$$

where

$$\sigma_2 = ab + bc + ca \qquad \sigma_3 = abc$$

Now we want to find the image of the hyperplanes under taking W invariants. First notice that L_1, L_2, and L_3 all map to a point. The hyperplanes are the vanishing locus of the H_{ij}, but in the a, b, c basis, the hyperplanes are instead where $a = b$ or $b = c$ or $c = a$. Then the claim is that the image of these hyperplanes is given by some equation of order 3 in σ_2 and order 2 in σ_3, i.e. $\left\{c_2\sigma_2^3 + c_3\sigma_3^2\right\}$ for some c_2 and c_3. In particular for $h = (a-b)(b-c)(c-a)$ we have that h^2 is exactly the equation cutting out this cusp.

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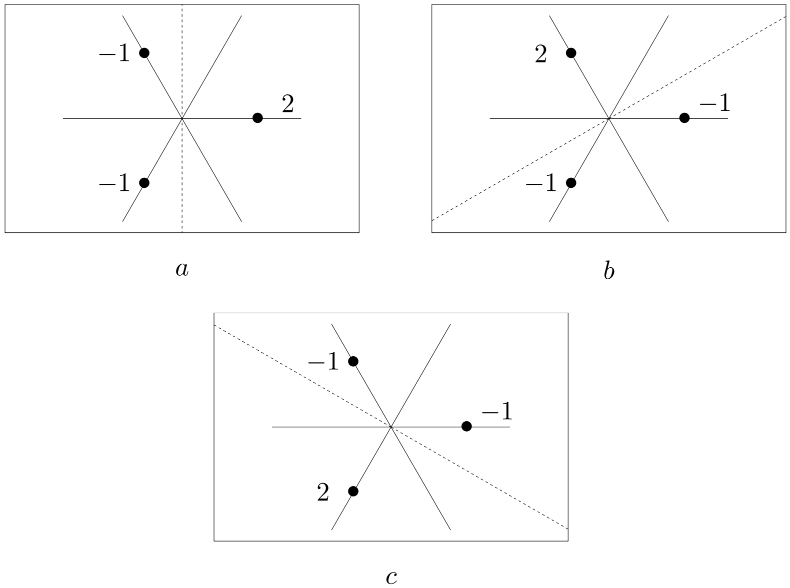

FIGURE 1. The values taken by our new basis a, b, and c. The functions vanish along the dotted hyperplanes now rather than the hyperplanes that the Weyl group reflects over.

2. Isomorphism of $\mathcal{U}\mathfrak{g}$ and $\operatorname{Sym}\mathfrak{g}$ as vector spaces but NOT representations

If we choose a PBW basis, we can get an identification $\mathcal{U}\mathfrak{g} \simeq \operatorname{Sym}(\mathfrak{g})$ as vector spaces, but certainly not as G-representations. The following example shows this.

Example 1. Take $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ with the usual PBW basis. If we look at $XY \in \mathcal{U}\mathfrak{g}$, then under this isomorphism with $\operatorname{Sym}\mathfrak{g}$ this element $XY \mapsto XY \in \operatorname{Sym}\mathfrak{g}$. But if we instead take $YX \in \mathcal{U}\mathfrak{g}$, the prescription is to rewrite this as $YX = XY + [Y, X] = XY - H$ which is in "PBW form" so this gets mapped to $XY - H \in \operatorname{Sym}\mathfrak{g}$. This is exactly the point of PBW, it is somehow telling you how to break symmetry.

Consider

$$g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C}) \ .$$

Then we're hoping that if we conjugate YX and map it to $\operatorname{Sym}\mathfrak{g}$, this is the same as mapping it and then conjugating it. We can calculate the action of Ad_g to be

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -a & -c \\ -b & a \end{pmatrix}$$

which means

$$H \mapsto -H \qquad X \mapsto -Y \qquad Y \mapsto -X$$

and therefore we have

$$\mathrm{Ad}_g(YX) = \mathrm{Ad}_g(Y)\,\mathrm{Ad}_g(X) = (-X)(-Y) = XY \ .$$

Therefore we have seen that if we first conjugate and then map to $\operatorname{Sym}\mathfrak{g}$ versus mapping to $\operatorname{Sym}\mathfrak{g}$ and then conjugating, we don't get the same result:

$$\begin{tikzcd} YX \arrow[r, mapsto] \arrow[d, mapsto, "\mathrm{Ad}_g"] & XY - H \arrow[drr, mapsto] & & \\ XY \arrow[r, mapsto, "\mathrm{PBW}"] & XY & \neq & YX + H \end{tikzcd}$$

Therefore these things are not isomorphic as G representations in this way.

3. Isomorphism of $\mathcal{U}\mathfrak{g}$ and $\operatorname{Sym}\mathfrak{g}$ as adjoint G representations

The goal is now to construct an isomorphism of these as representations. We will not be using PBW at all. We know $\mathcal{U}\mathfrak{g}$ is filtered, and $\operatorname{Sym}\mathfrak{g}$ is even graded, so it's certainly filtered.

Claim 2 (Good news)**.** There exists an isomorphism of adjoint G representations so that in particular, for any piece of our filtration of $\mathcal{U}\mathfrak{g}$ we have the following isomorphism:

$$\begin{tikzcd} \mathcal{U}\mathfrak{g} \arrow[r, "\sim"] & \operatorname{Sym}\mathfrak{g} \\ F^k\mathcal{U}\mathfrak{g} \arrow[u, hook] \arrow[r, "\sim"] & \bigoplus_{i=0}^{k} \operatorname{Sym}^i\mathfrak{g} \arrow[u, hook] \end{tikzcd}$$

Proof. We will prove this by induction on the filtration. The base case is just:

$$F^0\mathcal{U}\mathfrak{g} \simeq \mathbb{C} \simeq \operatorname{Sym}^0(\mathfrak{g})$$

So now suppose we have

$$F^{k-1}\mathcal{U}\mathfrak{g} \simeq \bigoplus_{i=0}^{k-1} \operatorname{Sym}^i(\mathfrak{g})$$

as G representations. Consider the following SESs:

$$\begin{tikzcd} 0 \arrow[r] & F^{k-1}\mathcal{U}\mathfrak{g} \arrow[r] & F^k\mathcal{U}\mathfrak{g} \arrow[r] & \mathrm{Gr}_k\,\mathcal{U}\mathfrak{g} \arrow[r] & 0 \\ 0 \arrow[r] & \bigoplus_{i=0}^{k-1} \operatorname{Sym}^i(\mathfrak{g}) \arrow[r] \arrow[u, "\sim"] & \bigoplus_{i=0}^{k} \operatorname{Sym}^i(\mathfrak{g}) \arrow[r] \arrow[u, "?"] & \operatorname{Sym}^k(\mathfrak{g}) \arrow[r] \arrow[u, "\sim"'] \arrow[l, bend right] & 0 \end{tikzcd}$$

where the bottom sequence naturally splits. Then since the category is semi-simple, the top SES splits. $\square$

4. More discussion of Harish Chandra

We won't be explicitly proving the HC isomorphism, but we will discuss it more so we at least feel comfortable with it. Recall the content of the theorem is that

$$\mathfrak{z}(\mathcal{U}\mathfrak{g}) \simeq (\mathcal{U}\mathfrak{h})^{W,\tilde{\cdot}}.$$

Geometrically, we can think of G/N as having a left action of G by left multiplication, and a right action of H which commutes with this action since

$$(gN)\,h = gh\left(h^{-1}Nh\right) = ghN$$

since H normalizes N.

Example 2. Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$. Then $G/N = \mathbb{C}^2 \setminus \{0\}$ and

$$gN \mapsto g \begin{pmatrix} 1 \\ 0 \end{pmatrix} .$$

Recall that $N = \left\langle \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \right\rangle$ is exactly the stabilizer of this vector $(1, 0)$. Now we can think about what this action does to vectors in $\mathbb{C}^2 \setminus \{0\}$. Take $h = \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$ inside the Cartan, and now we want to see what $gNh = ghN$ goes to:

$$ghN \mapsto gh \begin{pmatrix} 1 \\ 0 \end{pmatrix} = g \begin{pmatrix} z \\ 0 \end{pmatrix}$$

so $H \simeq \mathbb{C}^\times$ acts by dilation, and $\mathrm{SL}(2, \mathbb{C})$ acts as usual by linear transformations. So we have these two commuting actions on $\mathbb{C}^2 \setminus \{0\}$.

Exercise 1. Show than when we have these two commuting actions $G \circlearrowright G/N \circlearrowleft H$, then G-equivariant automorphisms of G/N are exactly just H acting on the right.

Solution. Since $B = N_G(N)$, we have $H \simeq B/N$. This is just a general fact that in any subgroup if you ask what are the G-equivariant automorphisms of the homogeneous space, the answer will be the normalizer modulo the stabilizer which in this case is H.

The point is that one only needs to know where one point goes since it is G equivariant.

Remark 1. This is very similar to when we were talking about highest weights.

Corollary 1. *G invariant vector fields on G/N are given by vector fields coming from $\mathfrak{h}$.*

Corollary 2 (More generally)**.** *The collection of G invariant differential operators on G/N is isomorphic to differential operators coming from $\mathcal{U}\mathfrak{h}$.*

Example 3. Let's return to $G = \mathrm{SL}(2, \mathbb{C})$ to see what's going on here. In this context this is saying that:

$$\mathrm{Aut}^G\left(\mathbb{C}^2 \setminus \{0\}\right) \simeq \mathbb{C}^\times$$

where $\mathbb{C}^\times$ acts by dilation. So if you need to map a vector to another vector in a way that is G-compatible, i.e. it commutes with linear tranformations, then the only way to do it is by dilation.

4.1. **Application.** The reason this abstract discussion is useful, is the following application. We can map

$$\mathfrak{z}(\mathcal{U}\mathfrak{g}) \to \mathrm{Diff}^G(G/N)$$

but these must come from $\mathcal{U}\mathfrak{h}$. In other words we have a factorization

$$\begin{tikzcd} \mathfrak{z}\mathcal{U}\mathfrak{g} \arrow[rr] \arrow[dr] & & \mathrm{Diff}^G(G/N) \\ & \mathcal{U}\mathfrak{h} \arrow[ur, dashed] & \end{tikzcd}$$

and this is the Harish Chandra homomorphism. In this language the theorem is saying that the image is actually $(\mathcal{U}\mathfrak{h})^{(W, \tilde{\cdot})}$.

To be continued... [1]

[1] A fire alarm went off at this point. Probably because the large amount of smoke in the air from the forest fires leaked into the building and set the alarms off.

LECTURE 23
MATH 261A

LECTURES BY: PROFESSOR DAVID NADLER
NOTES BY: JACKSON VAN DYKE

We will have lecture this week and Tuesday of next week. There will be office hours this week and next week as usual. The final will be posted this week, and it will be due on Monday December 10. The topic for the remaining lectures will be D-modules and Beilinson-Bernstein localization.

1. Harish-Chandra

For any complex simple Lie group G, we have the actions $G \circlearrowright G/N \circlearrowleft B/N \simeq H$ where G/N is the fundamental affine space.

Example 1. For $G = \mathrm{SL}(2, \mathbb{C})$, the fundamental affine space is $\mathbb{C}^2 \setminus \{0\}$, where $\mathrm{SL}(2, \mathbb{C})$ acts by fractional linear transformations, and $\mathbb{C}^\times \simeq H$ acts on the right by dilations.

We know we can map $\mathfrak{g} \to (G/N) \leftarrow \mathfrak{h}$ and extend this to $\mathcal{U}\mathfrak{g} \to \mathrm{Diff}(G/B) \leftarrow \mathcal{U}\mathfrak{h}$.

Theorem 1. *$\mathfrak{z}(\mathcal{U}\mathfrak{g}) \hookrightarrow \mathrm{Diff}(G/N)$ with image*

$$\mathrm{Diff}(G/N) \hookleftarrow (\mathcal{U}\mathfrak{h})^{(W,\tilde{\cdot})} \simeq (\mathrm{Sym}\,\mathfrak{h})^{(W,\tilde{\cdot})} \simeq \mathbb{C}[\mathfrak{h}^*]^{(W,\tilde{\cdot})}$$

so

$$\mathfrak{z}(\mathcal{U}\mathfrak{g}) \xrightarrow{\sim} \mathbb{C}\left[\mathfrak{h}^\times\right]^{(W,\tilde{\cdot})}$$

Example 2. $\mathfrak{z}(\mathcal{U}\mathfrak{sl}(2, \mathbb{C})) \simeq \mathbb{C}[K]$ where $K = H^2 + 2(XY + YX)$ is the Casimir. We should then think of its image under this isomorphism, $(H+1)^2 \in \mathbb{C}[\mathfrak{h}^\times]^{(W,\tilde{\cdot})}$, as the quadratic function that vanishes to order 2 at -1.

2. Beilinson-Bernstein localization

Let $G \circlearrowright G/B$ be a simple complex Lie group acting on this flag variety. As usual we can map $\mathfrak{g} \to \mathrm{Vect}(G/B)$ and $\mathcal{U}\mathfrak{g} \to \mathrm{Diff}(G/B)$.

2.1. **Algebraic vector fields and differential operators.** We want to restrict our attention to algebraic vector fields and algebraic differential operators. So let's see what those things are. G/B can be covered by affine spaces $\mathbb{C}^d$ where $d = \dim(G/B)$. We can do this by taking a coordinate flag $E_w^\bullet$ for $w \in W$. Recall the standard flag is:

$$E_{\mathrm{std}}^\bullet = \{\langle 0 \rangle \subset \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \cdots \mathbb{C}^n\}$$

and then

$$E_2^\bullet = w(E_{\mathrm{std}}^\bullet) = \left\{\langle 0 \rangle \subset \left\langle e_{w(1)} \right\rangle \subset \left\langle e_{w(1)}, e_{w(2)} \right\rangle \subset \cdots \right\} \ .$$

Date: November 27, 2018.

So given any coordinate flag we can define the affine space

$$A_w = \{E^\bullet \pitchfork E_w^\bullet\}$$

which means $E^k \pitchfork E_w^{n-k}$ for all k.

Exercise 1. Show that each of these $A_w \simeq \mathbb{C}^d$.

Now "algebraic" means that we only allow polynomial functions on the coordinate patches. So algebraic vector fields are vector fields such that on any coordinate patch, it will look like a polynomial function times $\partial / \partial x_i$ rather than a generic complex analytic function times these $\partial / \partial x_i$. Explicitly they are of the form:

$$x = \sum_{i=1}^{d} p_i(x)\, \partial_{x_i}$$

where the p_i are polynomials. We will write $\mathrm{Vect}^{\mathrm{alg}}(G/B)$ and $\mathrm{Diff}^{\mathrm{alg}}(G/B)$ for the algebraic vector fields and algebraic differential operators respectively.

Exercise 2. Show that the maps $\mathfrak{g} \to \mathrm{Vect}(G/B)$ and $\mathcal{U}\mathfrak{g} \to \mathrm{Diff}(G/B)$ land in algebraic vector fields and algebraic differential operators.

Example 3. For $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$, in one of the coordinate patches we saw that $X \mapsto \partial_x$, $H \mapsto -2x\partial_x$, and $Y \mapsto -x^2 \partial_x$ which are of course algebraic.

2.2. **A fundamental theorem.**

Theorem 2. *The map* $\mathcal{U}_0\mathfrak{g} \to \mathrm{Diff}^{alg}(G/B)$ *is an isomorphism, where we define*

$$\mathcal{U}_0\mathfrak{g} = \mathcal{U}\mathfrak{g} / \mathfrak{z}^0(\mathcal{U}\mathfrak{g})$$

where $\mathfrak{z}^0(\mathcal{U}\mathfrak{g})$ *is the augmentation ideal, i.e. the ideal of* $\mathfrak{z}(\mathcal{U}\mathfrak{g}) \simeq \mathbb{C}[\mathfrak{h}^\times]^{(W,\tilde{\cdot})}$ *vanishing at* $0 \in \mathfrak{h}^\times$.

This is compatible with the HC isomorphism in the following sense. Recall that when discussing HC we mapped $\mathfrak{g} \to \mathrm{Vect}^{\mathrm{alg}}(G/N) \leftarrow \mathfrak{h}$ and $\mathcal{U}\mathfrak{g} \to \mathrm{Diff}^{\mathrm{alg}}(G/N) \leftarrow \mathcal{U}\mathfrak{h}$. But now we can obtain $\mathrm{Diff}^{\mathrm{alg}}(G/B)$ from $\mathrm{Diff}^{\mathrm{alg}}(G/N)$ by doing "quantum Hamiltonian reduction." In particular we following the steps:

(1) Take H invariant differential operations on G/N, $\mathrm{Diff}^{\mathrm{alg}}(G/N)^H$.
(2) Quotient by $\mathcal{U}^0\mathfrak{h}$, which is the H-invariant differential operators along the fibers. Note that $\mathcal{U}^0\mathfrak{h}$ is the kernel of the map $\mathcal{U}^0\mathfrak{h} \to \mathcal{U}\mathfrak{h} \simeq \mathbb{C}[\mathfrak{h}^\times] \xrightarrow{\mathrm{ev}_0} \mathbb{C}$.

So in the end we get:

$$\mathrm{Diff}^{\mathrm{alg}}(G/B) = \mathrm{Diff}^{\mathrm{alg}}(G/N)^H / \mathcal{U}^0\mathfrak{h}$$

So the map $\mathcal{U}\mathfrak{g} \to \mathrm{Diff}^{\mathrm{alg}}(G/B)$ certainly must send the ideal $\mathfrak{z}^0(\mathcal{U}\mathfrak{g}) \subseteq \mathcal{U}^0\mathfrak{h}$ to 0.

Remark 1. Similarly we can prove that $\mathcal{U}_\lambda\mathfrak{g} \xrightarrow{\sim} \mathrm{Diff}_\lambda^{\mathrm{alg}}(G/B)$ where $\mathcal{U}_\lambda\mathfrak{g} = \mathcal{U}\mathfrak{g}/\mathfrak{z}^\lambda(\mathcal{U}\mathfrak{g})$, and

$$\mathrm{Diff}_\lambda^{\mathrm{alg}}(G/B) = \mathrm{Diff}^{\mathrm{alg}}(G/N)^H / \mathcal{U}^\lambda\mathfrak{h}\ .$$

This new object consists of what are called "twisted differential operators." We can think of this as taking the fiber of the moment map at λ rather than at 0. The most general version of this theorem that one might consider is:

Theorem 3. $\mathcal{U}\mathfrak{g} \otimes_{\mathfrak{z}\mathfrak{g}} \mathcal{U}\mathfrak{h} \xrightarrow{\sim} \mathrm{Diff}^{alg}(G/N)^H$

So in this case we somehow skip the second step of the Hamiltonian reduction above.

The point is that if we want to understand $\mathcal{U}\mathfrak{g}$ modules, or in particular irreducible $\mathcal{U}\mathfrak{g}$ modules, each of them will have a fixed central character, so each of them will come with some $\mathcal{U}_\lambda\mathfrak{g}$. And this theorem is telling us that the theory of $\mathcal{U}_\lambda\mathfrak{g}$ modules is the same as the theory of modules over the differential operators, so something very geometric. We will spend the next week or so talking about what it means to be a module over differential operators.

3. Modules over differential operators

We could tell this whole story for twisted differential operators, which is necessary to understand all $\mathcal{U}\mathfrak{g}$ modules, since studying differential operators only tells us about $\mathcal{U}_0\mathfrak{g}$. But we will keep it simple and just study $\mathrm{Diff}^{\mathrm{alg}}(G/B)$. Now we have the following key idea:

Key idea: To obtain modules over global differential operators from local modules over differential operators by taking global sections.

The point here is that we will construct modules by gluing together "local" modules. We couldn't do this in $\mathcal{U}_0\mathfrak{g}$ itself, but in G/B we can since

$$G/B = \bigcup_{w\in W} A_w$$

is just the union of affine pieces which are each $\mathbb{C}^d$ for $d = \dim G/B$. In particular, we will study differential operators on these pieces A_w and then glue them all together.

3.1. Local story: algebraic differential operators and their modules on $\mathbb{C}^d$.

Exercise 3. Show that the algebraic differential operators are exactly the Weyl algebra:

$$\mathrm{Diff}^{\mathrm{alg}}\left(\mathbb{C}^d\right) \simeq \mathbb{C}\left\langle x_1, \cdots, x_d, \partial_{x_1}, \cdots, \partial_{x_d}\right\rangle$$

where the x_i all commute, the ∂_{x_i} commute, and then

$$[\partial_{x_i}, x_j] = \partial_{x_i}x_j - x_j\partial_{x_i} = \begin{cases} 0 & i \neq j \\ 1 + x_i\partial_{x_i} - x_i\partial_{x_i} = 1 & i = j \end{cases}.$$

Now we will think about what modules are like over this algebra. First we will focus on the case $d = 1$ and do some examples. So in this case, $\mathrm{Diff}^{\mathrm{alg}}(\mathbb{C}) \simeq \mathbb{C}\langle x, \partial_x\rangle$.

Example 4. The free module $\mathbb{C}\langle x, \partial_x\rangle^{\oplus n}$ is of course a module.

Example 5. Polynomial functions $\mathcal{O}^{\mathrm{alg}}(\mathbb{C}) \simeq \mathbb{C}[x]$ is a module over $\mathrm{Diff}^{\mathrm{alg}}(\mathbb{C})$ where x acts as x and ∂_x differentiates x. Note that $\mathbb{C}\langle x, \partial_x\rangle / (\partial_x) \xrightarrow{\sim} \mathbb{C}[x]$ where $1 \mapsto 1$. This isn't free, but it's still nice. This is a somehow small module since it's the quotient of a rank-one free module.

Example 6. We can also consider the rational functions

$$\mathcal{K}^{\mathrm{alg}}(\mathbb{C}) \simeq \mathbb{C}(x) = \left\{\frac{p}{q} \mid q \not\equiv 0\right\}.$$

This is also a module, but it is not finitely generated, because whenever you think you've finitely generated it, some function walks into the room with a deeper pole.

Example 7. We can consider

$$\mathcal{O}^{\mathrm{alg}}(\mathbb{C})\langle e^x\rangle = \{p \cdot e^x \mid p \text{ polynomial}\}$$

but we have to check that

$$\partial_x (pe^x) = p'e^x + pe^x = (p' + p)\, e^x \ .$$

This is somehow a small module, so we expect it to be somehow surjected upon by a rank 1 free module. Indeed, $\mathbb{C}\langle x, \partial_x\rangle / (\partial_x - 1) \xrightarrow{\sim} \mathcal{O}^{\mathrm{alg}}(\mathbb{C})\langle e^x\rangle$ where we map $1 \mapsto e^x$.

Example 8. We can also consider any sufficiently differentiable function space on $\mathbb{C}$, but these are somehow huge and not so algebraic.

Now we might wonder if there is a module M with $\dim_{\mathbb{C}} M < \infty$. Certainly none of them so far have satisfied this. An easier question to ask might be to forget about ∂_x, and think of finite dimensional $\mathbb{C}[x]$ modules.

Exercise 4. Show that any finite dimensional $\mathbb{C}[x]$ module is a finite dimensional vector spaces equipped with an endomorphism. Show that

$$V = \bigoplus_{i=1}^{k} \mathbb{C}[x] / (x - \lambda_i)^{d_i} \ .$$

Note that each of these is a Jordan block.

So now we need to ask ourselves how to add ∂_x into the picture and act on such a V.

Claim 1. Finite dimensional $\mathbb{C}\langle x, \partial_x\rangle$-modules M are all trivial.

First we need to check how ∂_x must act on an eigenvector, then we just need to notice what happens when we keep applying ∂_x.

Remark 2. We can also show it more algebraically. If we have some finite dimensional module M, then we can represent x and ∂_x as two matrices A and B which act as linear operators on M. Then since $[x, \partial_x] = 1$, $[A, B] = I$, but if M is finite dimensional we have a well defined Tr, so

$$n = \mathrm{Tr}(I) = \mathrm{Tr}(AB - BA) = 0$$

so M must be zero dimensional, or Tr is not well defined.

LECTURE 24
MATH 261A

LECTURES BY: PROFESSOR DAVID NADLER
NOTES BY: JACKSON VAN DYKE

Recall from last time we stated the following theorem:

Theorem 1. *For $\mathfrak{g}$ a semisimple complex Lie algebra, we have an isomorphism:*

$$\mathcal{U}_0\mathfrak{g} \xrightarrow{\sim} \mathrm{Diff}^{alg}(G/B)$$

Remark 1. The G action on G/B leads to a map $\mathfrak{g} \to \mathrm{Vect}^{\mathrm{alg}}(G/B)$, which leads to a map $\mathcal{U}\mathfrak{g} \to \mathrm{Diff}^{\mathrm{alg}}(G/B)$. Last time we discussed why this factors through $\mathcal{U}\mathfrak{g}$ to $\mathcal{U}_0\mathfrak{g} = \mathcal{U}\mathfrak{g}/\mathfrak{z}^0\mathfrak{g}$, where $\mathfrak{z}^0\mathfrak{g}$ is the ideal I_0 under the identification $\mathfrak{z}\mathfrak{g} \simeq \mathbb{C}[\mathfrak{h}^\times]^{(W,\tilde{\cdot})}$. And from the same argument, is then injective.

1. Why is this map surjective

We will now try to see why we should expect this map to be surjective.

Remark 2. If this is indeed surjective, we might wonder what hits the functions on the right, but the answer is that they're all constant, since G/B is compact. The example to keep in mind is $\mathbb{P}^1$.

Remark 3. If we're somehow only concerned with representation theory of $\mathfrak{g}$, we don't really need this to be surjective, but we will make this comment anyway.

First note that the above maps are actually maps of filtered algebras. I.e.

$$\mathcal{U}\mathfrak{g} \to \mathcal{U}_0\mathfrak{g} \to \mathrm{Diff}^{\mathrm{alg}}(G/B)$$

respect the natural filtrations on these objects. The first has the tensor algebra filtration, the second has the tensor algebra filtration modulo some filtered ideal, and the filtration on $\mathrm{Diff}^{\mathrm{alg}}(G/B)$ is given by the order of a given differential operator. These all somehow look like

$$\sum_\alpha p_\alpha(x)\,\partial_x^\alpha$$

where $p_\alpha(x)$ are polynomials where for big enough α, they're all zero. Then the filtration is given by the order of α.

Date: November 29, 2018.

Now we can pass to the associated graded algebras:

$$\begin{tikzcd}
\mathrm{Sym}(\mathfrak{g}) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Sym}(\mathfrak{g}) / \left(\mathrm{Sym}(\mathfrak{g})^G\right)^0 \simeq \mathbb{C}[\mathfrak{g}_{\chi=0}] \arrow[r] \arrow[d, "\simeq"] & \mathrm{Gr}\,\mathrm{Diff}^{\mathrm{alg}}(G/B) \\
\mathbb{C}[\mathfrak{g}^*] & \left(\mathbb{C}[\mathfrak{g}^*]^G\right)^0 \arrow[d, "\simeq"] & \\
 & \left(\mathbb{C}[\mathfrak{h}^*]^W\right)^0 = I^0 &
\end{tikzcd}$$

where $\left(\mathrm{Sym}(\mathfrak{g})^G\right)^0$ is as follows. Recall we wrote down an isomorphism of G-representations between $\mathcal{U}\mathfrak{g}$ and $\mathrm{Sym}\,\mathfrak{g}$. And under this isomorphism, the center goes to the G-invariant piece. Then we take the ideal of zero inside of this, i.e. the ideal of things which vanish when we just look at their constant piece. More precisely,

$$\begin{aligned}
\mathrm{Sym}(\mathfrak{g}) &\simeq \mathbb{C} \oplus \mathfrak{g} \oplus \mathrm{Sym}^2 \mathfrak{g} \oplus \cdots \\
\mathrm{Sym}(\mathfrak{g})^G &\simeq \mathbb{C} \oplus \langle 0 \rangle \oplus \mathrm{Sym}^2 \mathfrak{g} \oplus \cdots \\
\left(\mathrm{Sym}(\mathfrak{g})^G\right)^0 &\simeq \langle 0 \rangle \oplus \langle 0 \rangle \oplus \mathrm{Sym}^2 \mathfrak{g} \oplus \cdots
\end{aligned}$$

We can think of this quotient by $\left(\mathrm{Sym}(\mathfrak{g})^G\right)^0$ as being functions $\mathbb{C}[\mathfrak{g}_{\chi=0}]$ where $\chi : \mathfrak{g}^* \to \mathfrak{h}^*$ is the characteristic polynomial map, and $\mathcal{N} = \mathfrak{g}^*_{\chi=0}$ consists of the matrices whose eigenvalues are all 0. The sort of geometric picture is:

$$\begin{tikzcd}
\mathfrak{g}^* \arrow[d, "\chi"] & \mathfrak{g}^*_{\chi=0} = \mathcal{N} \arrow[l, hook'] \arrow[d] \\
\mathfrak{h}^*/W & 0 \arrow[l, phantom, "\ni"]
\end{tikzcd}$$

Note that this diagram is the fiber in this category.

So on associated graded algebras, $\mathcal{U}\mathfrak{g}$ becomes functions on $\mathfrak{g}^*$, and $\mathcal{U}^0\mathfrak{g}$ becomes functions on $\mathcal{N}$. In other words, if we pass to algebraic functions, we get

$$\begin{tikzcd}
\mathbb{C}[\mathfrak{g}^*] \arrow[r] \arrow[d, equal] & \mathbb{C}[\mathcal{N}] \arrow[d, equal] \\
\mathrm{Sym}(\mathfrak{g}) \arrow[r] & \mathrm{Sym}(\mathfrak{g}) / \left(\mathrm{Sym}(\mathfrak{g})^G\right)^0 \\
\mathbb{C}[\mathfrak{h}^*]^W \arrow[u, "\chi^*"] \arrow[r, "\mathrm{res}_0"] & \mathbb{C} \arrow[u]
\end{tikzcd}$$

So on the level of associated graded algebras, when we restrict to $\mathcal{U}_0\mathfrak{g}$, instead of looking at functions on all of $\mathfrak{g}^*$, we're just looking at functions on $\mathcal{N}$. Now if we quantize, we get

$$\begin{array}{ccc} \mathcal{U}\mathfrak{g} & \longrightarrow & \mathcal{U}_0\mathfrak{g} \\ \uparrow & & \uparrow \\ \mathfrak{z}\mathfrak{g} & \longrightarrow & \mathbb{C} \end{array}$$

Remark 4. We have seen that when we pass to associated graded, $\mathcal{U}_0\mathfrak{g}$ goes to functions on $\mathcal{N}$. For $\mathcal{U}_\lambda\mathfrak{g}$, this goes to functions on the fiber above λ as in the above diagram.

Now finally, the associated graded algebra of $\mathrm{Diff}^{\mathrm{alg}}(G/B)$ is

$$\mathrm{Gr}\,\mathrm{Diff}^{\mathrm{alg}}(G/B) \simeq \mathcal{O}^{\mathrm{alg}}(T^*G/B)\ .$$

This somehow has nothing to do with G/B, this is just a general fact. This is a canonical construction.

Remark 5. In a kind of naive way, we can say that any time we have ∂_x, we replace it by the functions linear along the fiber, which to a covector, pairs with ∂_x.

Remark 6. Certinaly every vector field gives you a function on the cotangent bundle where you just pair with the vector field. Now if all of those commute, then you get all functions on the cotangent bundle, and if they don't, then you're back to thinking about differential operators.

To convince ourselves of this, we can consider the following local picture. Recall that locally, differential operators on some $\mathbb{C}^d$, are $\mathrm{Diff}^{\mathrm{alg}}\left(\mathbb{C}^d\right) = \mathbb{C}\left[x_1, .., x_d, \partial_{x_1}, \cdots, \partial_{x_d}\right]$. Then the filtration is given by the order of α. In particular,

$$\mathrm{Diff}^{\leq 0}\left(\mathbb{C}^d\right) = \mathcal{O}\left(\mathbb{C}^d\right) \subset \mathrm{Diff}^1\left(\mathbb{C}^d\right) = \mathcal{O}\left(\mathbb{C}^d\right)\langle \partial_{x_1}, \cdots, \partial_{x_d}\rangle \subset \cdots$$

and in general

$$\mathrm{Diff}^{\leq n}\left(\mathbb{C}^d\right) = \mathrm{Diff}^{\leq (n-1)}\left(\mathbb{C}^d\right)\langle \partial_{x_1}, \cdots, \partial_{x_d}\rangle$$

Now when we pass to associated graded, we get

$$\begin{aligned} \mathrm{Gr}^0 &\simeq \mathcal{O}\left(\mathbb{C}^d\right) \\ \mathrm{Gr}^1 &\simeq \mathcal{O}\left(\mathbb{C}^d\right)\langle \xi_1, \cdots, \xi_d\rangle \\ \mathrm{Gr}^2 &\cong \mathcal{O}\left(\mathbb{C}^d\right)\langle \xi_i\xi_j \forall i, j\rangle \end{aligned}$$

where the ξ_i are fiber-wise linear functions on the cotangent bundle which act as $\xi_i(x, \eta) = \eta_i$ and the $\xi_i\xi_j$ are fiber-wise quadratic functions which act as $\xi_i\xi_j(x, \eta) = \eta_i\eta_j$. Now we can just glue this local picture together.

Now we want to understand the last map $\mathcal{U}_0\mathfrak{g} \to \mathrm{Diff}^{\mathrm{alg}}(G/B)$ on the level of associated graded algebras. Recall we've identified these with $\mathcal{O}^{\mathrm{alg}}(\mathcal{N})$ and $\mathcal{O}^{\mathrm{alg}}(T^*G/B)$ respectively. Then the claim is that the map $\mathcal{U}_0\mathfrak{g} \to \mathrm{Diff}^{\mathrm{alg}}(G/B)$ is an isomorphism. In particular, we claim that this is an isomorphism iff it is an isomorphism on the level of the associated graded algebras, and that this is an isomorphism.

Exercise 1. The G action on G/B gives you a moment map $\mu : T^*G/B \to \mathfrak{g}^*$. Show:

(1) $\mathrm{im}(\mu)$ is $\mathcal{N} \subseteq \mathfrak{g}^*$
(2) μ^* is $\mathcal{O}^{\mathrm{alg}}(\mathcal{N}) \to \mathcal{O}^{\mathrm{alg}}(T^*G/B)$.

(3) μ is a resolution, in particular proper and surjective, and conclude μ^* is an isomorphism.

In summary, we have the following:[1]

$$\begin{tikzcd} \mathfrak{g}^* \arrow[d, "\text{quantize}"'] & \mathcal{N} \arrow[l, hook'] & T^*G/B \arrow[l, two heads] \\ \mathcal{U}\mathfrak{g} \arrow[r] & \mathcal{U}_0\mathfrak{g} \arrow[r, "\sim"] & \mathrm{Diff}^{\mathrm{alg}}(G/B) \end{tikzcd}$$

This is a very nice picture, because we have these geometric spaces, and then consider functions on them, and then we deform these functions to be non-commutative and we get this picture.

2. Local modules

Now we return to the question of modules over differential algebraic operators on $\mathbb{C}^d$. Recall

$$\mathrm{Diff}^{\mathrm{alg}}\left(\mathbb{C}^d\right) \simeq \mathbb{C}\left[x_1, \cdots, x_d, \partial_{x_1}, \cdots, \partial_{x_d}\right]$$

In particular, we were wondering if there are any finite dimensional representations over this. Suppose M is a finite dimensional representation, then x_i and ∂_{x_i} are just matrices and then there is a well defined trace, so $\operatorname{tr}([x_i, \partial_{x_i}]) = 0 = \operatorname{tr}(I) = n$. So the "smallest" modules are the size of the examples from last time which were all somehow like $\mathcal{O}\left(\mathbb{C}^d\right) \simeq \mathbb{C}[x_1, \cdots, x_d]$ and $\mathcal{O}\left(\mathbb{C}^d\right) e^{ax} \simeq e^{ax}\mathbb{C}[x_1, \cdots, x_d]$.

There are more of this sort when we consider the x_i and the ∂_{x_i} on the same footing. In particular, we have some Fourier transform symmetries $\mathrm{FT}_i : \mathrm{Diff}^{\mathrm{alg}}\left(\mathbb{C}^d\right) \to \mathrm{Diff}^{\mathrm{alg}}\left(\mathbb{C}^d\right)$ which map $x_j \mapsto x_j$ and $\partial_{x_j} \to \partial_{x_j}$ for $j \neq i$ and $x_i \mapsto \partial_{x_i}$ and $\partial_{x_i} \mapsto x_i$. So another example is:

$$\Delta(0) \simeq \mathbb{C}\left[\partial_{x_1}, \cdots, \partial_{x_d}\right] \simeq \mathrm{Diff}^{\mathrm{alg}}\left(\mathbb{C}^d\right) / (x_1, \cdots, x_d) \ .$$

This is like $\mathcal{O}\left(\mathbb{C}^d\right) \simeq \mathrm{Diff}^{\mathrm{alg}}\left(\mathbb{C}^d\right) / (\partial_{x_1}, \cdots, \partial_{x_d})$ except we instead quotient out by the x_i. We also have

$$\Delta(p) \simeq \mathbb{C}\left[\partial_{x_1}, \cdots, \partial_{x_d}\right] \simeq \mathrm{Diff}^{\mathrm{alg}}\left(\mathbb{C}^d\right) / (x_1 - p_1, \cdots, x_d - p_d) \ .$$

These are called the "delta functions" and can be thought of as distributions.

3. D-modules

Definition 1. A D-module M on G/B is a compatible collection of $\mathrm{Diff}^{\mathrm{alg}}(A_w)$-modules.

More precisely we want

$$M_w|_{A_w \cap A_{w'}} \simeq M_{w'}|_{A_w \cap A_{w'}}$$

given by $\varphi_w^{w'}$ and then there's a cocycle condition.

By restriction we mean the following. Given a $\mathrm{Diff}^{\mathrm{alg}}\left(\mathbb{C}^d\right)$-module M and an open $\mathcal{U} = \{p_1 \neq 0, \cdots, p_l \neq 0\} \subseteq \mathbb{C}^d$ then set

$$M|_U = M\left[\frac{1}{p_1}, \cdots, \frac{1}{p_l}\right]$$

[1] Professor Nadler says we should be screaming this in the middle of the night. He also says this is potentially tattoo worthy mathematics.

Theorem 2 (Beilinson-Bernstein localization)**.** $\mathcal{U}_0$-**Mod** *is equivalent to D-modules on G/B where we send a D-module M to its global sections* $\Gamma(G/B, M)$.

Note that we define the global sections to be the equalizer of the following diagram:

$$\Gamma(G/B, M) \longrightarrow \prod_{w\in W} M_w \rightrightarrows \prod_{w,w'} M_w|_{A_{w'}}$$

3.1. ***D*-modules on $\mathbb{P}^1$.** We will focus on $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$, so $G/B \simeq \mathbb{P}^1 \simeq A_1 \cup A_\sigma \simeq \mathbb{C} \cup \mathbb{C}$ where $W \simeq \mathbb{Z}/2 \simeq \{1, \sigma\}$. Recall that since $E^\bullet_{\text{std}} = \langle e_1 \rangle$, A_1 is the subset of lines l in $\mathbb{P}^1$ where l transverse to e_1, and then $E^\bullet_\sigma = \langle e_2 \rangle$, so A_σ is the subset of lines l in $\mathbb{P}^1$ which are transverse to e_2. A_1 has coordinate $t = s^{-1}$, and A_σ has coordinate s, where s is the slope of the line, i.e. the intersection with the line at $x = 1$.

Therefore a D-module M has two parts, it is a pair M_1 and M_σ which are modules over $\mathbb{C}[t, \partial_t]$ and $\mathbb{C}[s, \partial_s]$ respectively. They also come with an isomorphism

$$M_1\left[t^{-1}\right] \simeq M_\sigma\left[s^{-1}\right] =: M_{1,\sigma}$$

as modules over $\mathbb{C}\left[t, t^{-1}, \partial_t\right] \simeq \mathbb{C}\left[s, s^{-1}, \partial_s\right]$.

The global sections are:

$$\Gamma\left(\mathbb{P}^1, M\right) = \ker\left[M_1 \times M_\sigma \to M_{1,\sigma}\right]$$

so these are pairs which map to $m_1, m_\sigma \mapsto m_1 - m_\sigma$. Recall $\mathfrak{sl}(2,\mathbb{C}) \to \text{Vect}^{\text{alg}}\left(\mathbb{P}^1\right)$.

Example 1. $\mathcal{O}^{\text{alg}}\left(\mathbb{P}^1\right) = \Gamma\left(\mathbb{P}^1, \mathcal{O}\right)$ for some D-module $\mathcal{O}$, defined by $\mathcal{O}(A_1) = \mathbb{C}[t]$ and $\mathcal{O}(A_\sigma) = \mathbb{C}[s]$. So this is saying all polynomials on a compact space are constant. This is a complicated way of telling you the trivial representation of $\mathfrak{sl}(2,\mathbb{C})$.

Example 2. The D-module $\Delta_{s=0}$ at $s = 0$ is given by the pieces $\Delta_{s=0}(A_1) = \langle 0 \rangle$, and $\Delta_{s=0}(A_\sigma) = \mathbb{C}[s, \partial_s]/(s)$.

Exercise 2. What representation of $\mathfrak{sl}(2,\mathbb{C})$ does this correspond to?

The BGG resolution will come from the Schubert cells which we will see on Tuesday.

LECTURE 25
MATH 261A

LECTURES BY: PROFESSOR DAVID NADLER
NOTES BY: JACKSON VAN DYKE

Today we will pick up where we left off with $\mathfrak{sl}(2,\mathbb{C})$ and return to the BGG resolution.

1. D-modules

Recall we were considering $G = \mathrm{SL}(2,\mathbb{C}) \circlearrowright \mathbb{P}^1 \simeq \mathcal{B} \simeq G/B$. This is the usual action by fractional linear transformations. So we get a map $\mathfrak{g} \to \mathrm{Vect}(\mathbb{P}^1)$. Then we have our favorite elements X, H, and Y, which go to

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto s^2\partial_s \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto 2s\partial_s \qquad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mapsto -\partial_s$$

in the chart $A_\sigma = \{l \pitchfork \langle e_2 \rangle\}$. Recall a D-module M on $\mathbb{P}^1$ is a compatible pair M_1 and M_σ where M_1 is a module over the Weyl algebra $\mathbb{C}\langle s, \partial_s\rangle$ and M_2 is a module over $\mathbb{C}\langle t, \partial_t\rangle$ where $t := s^{-1}$. Then these have to agree on the overlap. Equivalently, the following diagram commutes:

$$\begin{tikzcd} & \mathbb{C}\langle s,\partial_s\rangle \arrow[dr] & \\ \mathcal{U}\mathfrak{g} \arrow[ur] \arrow[dr] & & \mathbb{C}\left\langle s, s^{-1}, \partial_s\right\rangle = \mathbb{C}\left\langle t, t^{-1}, \partial_t\right\rangle \\ & \mathbb{C}\langle t,\partial_t\rangle \arrow[ur] & \end{tikzcd}$$

The global sections of this pair, $\Gamma(\mathbb{P}^1, M)$, comprise the kernel of the map:

$$M_1 \times M_\sigma \xrightarrow{\mathrm{res}_1 - \mathrm{res}_\sigma} M_1|_{A_\sigma} = M_\sigma|_{A_1}$$

Now because of the above compatibility, this is naturally a $\mathcal{U}\mathfrak{g}$ module. Now we want to match D-modules and $\mathcal{U}\mathfrak{g}$-modules.

One thing we could do is write down some D-modules and see what we get when we take global sections. We could also pick a representation and see some D-module that hits it. There are plenty of ways to play this game.

Example 1. Let's start with our favorite D-module $\mathcal{O}$, which consists of algebraic functions on $\mathbb{P}^1$. In the charts this is $\mathcal{O}_1 = \mathbb{C}[t]$ and $\mathcal{O}_\sigma = \mathbb{C}[s]$. Then the global sections $\Gamma(\mathbb{P}^1, \mathcal{O})$ are given by the kernel of $\mathbb{C}[t] \times \mathbb{C}[s] \to \mathbb{C}\left[t, t^{-1}\right]$ which takes $(f(t), g(s)) \mapsto f(t) - g\left(t^{-1}\right)$. But if this is 0, then $f = g$ is a constant, and this is indeed the trivial representation.

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Example 2. Fix $l = \langle e_1 \rangle$, and define $\Delta(l)$ to be the algebraic delta function at l. On charts, $\Delta(l)_1 = \langle 0 \rangle$, and $\Delta(l)_\sigma = \mathbb{C}[s, \partial_s] / (s) = \mathbb{C}[\partial_s]$. Note that we can write this as $\mathbb{C}[\partial_s] = \mathbb{C} \oplus \mathbb{C}\partial_s \oplus \mathbb{C}\partial_s^2 \oplus \cdots$ so 1 is playing the role of a distribution which takes a function and tells you the value at a point. Then ∂_s is playing the roll of the distribution which eats a functions, takes the derivative, and tells you the value of the derivative at a point, and so on. The global sections are

$$\Gamma\left(\mathbb{P}^1, \Delta(l)\right) = \mathbb{C}\langle s, \partial_s \rangle / \langle s \rangle \ .$$

Now to find which representation this is, we look at the image of the basis in $\operatorname{Vect}\left(\mathbb{P}^1\right)$, and act it on this. First notice that

$$X \cdot 1 = s^2 \partial_s \cdot 1 = s(\partial_s s - 1) \cdot 1$$

since we quotiented out by (s). Therefore 1 is an X highest-weight vector.

Now we want to see what the weight of the highest weight vector is by calculating:

$$H \cdot 1 = 2s\partial_s \cdot 1 = 2(\partial_s s - 1) \cdot 1 = -2 \cdot 1$$

so 1 is a highest weight vector of highest weight -2. We already saw Y generates this, so we have that ∂_s has weight -4, ∂_s^2 has weight -6, etc. This shows us that this is the Verma module $V_{-2} = \mathcal{U}\mathfrak{sl}(2,\mathbb{C}) \otimes_{\mathcal{U}\mathfrak{b}} \mathbb{C}_{-2}$.

Remark 1. If we change the pole of the delta function, i.e. we consider $\Delta(l)$ for some $l \in \mathbb{P}^1$, then the representation is a Verma module, but for a different choice of Borel.

Now let's pick a representation and try to construct a D-module M such that when we take the global sections $\Gamma\left(\mathbb{P}^1, M\right)$ we get this representation. Our favorite $\mathcal{U}\mathfrak{g}$ module with trivial central character, is $\mathcal{U}_0 = \mathcal{U}\mathfrak{sl}(2,\mathbb{C}) / \mathfrak{z}^0\mathfrak{sl}(2,\mathbb{C})$ where $\mathfrak{z}^0\mathfrak{sl}(2,\mathbb{C})$ is generated by the Casimir. As it turns out, if we take the D-module of differential operators $M = D$, where $D_1 = \mathbb{C}\langle t, \partial_t \rangle$ and $D_\sigma = \mathbb{C}\langle s, \partial_s \rangle$, then the global sections are indeed $\mathcal{U}_0\mathfrak{g}$.

Remark 2. The collection of global sections of a D-module is the same as

$$\Gamma\left(\mathbb{P}^1, M\right) = \operatorname{Hom}_{D\text{-}\mathbf{Mod}}(D, M)$$

where D is the free module. This is supposed to be like the story in algebra where we have some ring A and a module over A, then if we want to uncover the underlying abelian group structure of the module, we can just take Hom from the free rank 1 A-module to the module in question.

There is always a left adjoint to such a construction. If we have a representation V, we can tensor $D \otimes_{\mathcal{U}_0\mathfrak{sl}(2,\mathbb{C})} V$ where D is a D-module. We can form this tensor since $\mathcal{U}_0(\mathfrak{sl}(2,\mathbb{C}))$ maps to D. This is what's called localization. This is the usual adjunction between Hom and tensor, which in this case is an equality.

2. BGG Resolution

2.1. $\mathfrak{sl}(2,\mathbb{C})$. We first do this for $\mathfrak{sl}(2,\mathbb{C})$ and trivial central character. There is only one finite-dimensional representation with trivial central character, which is the trivial representation. Then we saw that $V_0 \twoheadrightarrow \mathbb{C}$, and in particular

$$0 \to V_{-2} \hookrightarrow V_0 \twoheadrightarrow \mathbb{C} \to 0$$

is exact. Now we want to try to understand what is going on with the D-modules. So we will localize to obtain a SES of D-modules:

$$0 \to \Delta(l) \hookrightarrow ?? \twoheadrightarrow \mathcal{O} \to 0$$

where $l = \langle e_1 \rangle$. As it turns out, "??" is:

$$\Delta(U_l) = D \otimes_{\mathcal{U}_0 \mathfrak{sl}(2,\mathbb{C})} V_0$$

which we will specify in each coordinate chart. On the chart A_1, this is just $\mathcal{O} = \mathbb{C}[t]$. On A_σ, this is $\mathbb{C}\langle s, \partial_s\rangle / (s\partial_s)$. We say this consists of the algebraic distributions on $\mathbb{P}^1 \setminus \{l\}$. Now we would need to check that when we invert s and t, these become the same.

On the A_σ chart the SES looks like

$$0 \to \mathbb{C}\langle s, \partial_s\rangle / (s) \hookrightarrow \mathbb{C}\langle s, \partial_s\rangle / (s\partial_s) \twoheadrightarrow \mathbb{C}\langle s, \partial_s\rangle / (\partial_s) \to 0$$

where $1 \mapsto \partial_s$ under the injection, and $1 \mapsto 1$ under the surjection. On A_1 the SES looks like

$$0 \to \langle 0\rangle \hookrightarrow \mathbb{C}[t] \twoheadrightarrow \mathbb{C}[t] \to 0$$

So to l we have assigned $\Delta(l)$, and to the rest, U_l, we assigned this $\Delta(U_l)$. So we cut $\mathbb{P}^1$ up into a point and its complement, and to each of these we have assigned a canonical module, which is the Verma module. And then the fact that the flag variety is a union of these pieces is manifested by functions on the flag variety being built up in a short exact sequence like this.

2.2. **General story.** In general, we have the Schubert stratification of the flag variety:

$$\mathcal{B} = G/B = \bigcup_{w \in W} \mathcal{B}_w$$

where $\mathcal{B}_w$ consists of the Borels $\mathfrak{b} \subseteq \mathfrak{g}$ such that the relative position[1] of $\mathfrak{b}$ w.r.t. fixed $\mathfrak{b}_0$ is w. Then to each $\mathcal{B}_w$ we can assign the D-module of algebraic distributions on $\mathcal{B}_w$, $\Delta(\mathcal{B}_w)$, and then the collection of global sections is a Verma for this Borel and for some character. And so the BGG resolution is the obvious fact that the D-module of functions can be cut up into distributions/functions on each of the Schubert pieces.

[1] Up to linear algebra, if one flag is standard, then the other can be made to be some coordinate flag, and w is which one.