LECTURE 1
MATH 261

LECTURES BY: PROFESSOR DAVID NADLER
NOTES BY: JACKSON VAN DYKE

The text is Varadarajan's Lie groups, Lie algebras and their representations. We won't follow this closely. Office hours haven't been specified yet.

1. Manifolds

1.1. **Preliminaries.** We will start with a review of smooth manifolds.

Definition 1. A manifold is a pair $\left(M, \{\mathcal{U}_\alpha, \varphi_\alpha\}_{\alpha \in A}\right)$ where M is a topological space, and $\{\mathcal{U}_\alpha, \varphi_\alpha\}$ is an atlas. This means the $\mathcal{U}_\alpha$ are open subsets and

$$\varphi_\alpha : \mathcal{U}_\alpha \xrightarrow{\sim} V_\alpha \subset \mathbb{R}^n$$

are homeomorphisms. These must satisfy the properties:

(1) The $\mathcal{U}_\alpha$ cover M.
(2) Compatibility in the sense that $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha \left(U_\alpha \cap U_\beta\right) \to \varphi_\beta \left(\mathcal{U}_\alpha \cap \mathcal{U}_\beta\right)$ is smooth.
(3) Maximality.

Example 1. Consider the n-sphere

$$S^n = \left\{ x \in \mathbb{R}^n \mid |x|^2 = 1 \right\}$$

We have effectively glued two copies of $\mathbb{R}^n$ according to open embeddings.

If we go about making spaces by gluing things together, we get two sort of pathologies. The first is that the resulting space will not always be Hausdorff.

Example 2. Consider $M = \mathbb{R} \amalg_{\mathbb{R}^\times} \mathbb{R}$. That is we take the disjoint union of two copies of $\mathbb{R}$ glued along the nonzero values in $\mathbb{R}$. This can be imagined as $\mathbb{R}$ with two zeros. The basic problem here is that continuous functions can't tell the difference between these two points.

Remark 1. Note that we can perform this gluing in two different ways. If we embed $\mathbb{R}^\times$ with the identity, we get this pathological non-Hausdorff space. If we however take one embedding to be the identity, and the other embedding to be the inverse function, then you get a different space. In fact this second space turns out to be S^1.

This motivates some typical additional assumptions:

(1) M is Hausdorff.
(2) M is paracompact.

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Paracompactness just means that every time we have an open cover, we can find a refinement of this cover such that the resulting open cover is locally finite. A typical example of what this condition restricts us from considering, i.e. something which is not paracompact, is the long line. The main reason that we like these conditions, is that we want well-behaved function theory. For example we want partitions of unity subordinate to covers.

For this class, all n-manifolds will be closed submanifolds of $\mathbb{R}^N$ for some $n \ll N$.

1.2. **Categorical point of view.** There is a category **Mfd** where the objects are manifolds, and between any two manifolds, we have the set $\operatorname{Hom}_{\mathbf{Mfd}}(M, N)$ which consists of the smooth maps $M \to N$. Note that we have compositions

$$\operatorname{Hom}(M, N) \times \operatorname{Hom}(N, P) \to \operatorname{Hom}(M, P)$$

given by the set-theoretic composition.

Note that there are many flavors:

(1) Smooth manifolds (smooth)
(2) Complex manifolds (holomorphic)
(3) Smooth algebraic varieties (polynomial maps)
(4) Banach manifolds (smooth)

The point is, there are lots of contexts where it makes sense to talk about a manifold, and these contexts are characterized by a particular notion of a "good" function.

2. Lie groups

Definition 2. A *Lie group* G is a group object in **Mfd**.

This means the following:

(1) G is a manifold
(2) G is a group
(3) These structures are compatible in the sense that multiplication $G \times G \xrightarrow{m} G$ and inverse $G \xrightarrow{i} G$ are smooth.

Recall that this means

(1) m is associative
(2) There is a unit $e \in G$ such that

$$m(g, e) = m(e, g) = g \qquad\qquad m(g, i(g)) = e = m(i(g), g)$$

Exercise 1. Derive that i is smooth from the fact that m is smooth.

Example 3. S^1 is a Lie group in the sense that

$$S^1 = \mathbb{R}/\mathbb{Z}$$

where the multiplication is just addition. Via an exponential map we can also write this as

$$\mathbb{R}/\mathbb{Z} = \{z \in \mathbb{Z} \mid |z| = 1\}$$

We can also regard $S^1 \hookrightarrow \mathbb{C}^\times$ as a subset of the nonzero complex numbers, which is also a Lie group. We say S^1 is a *Lie subgroup.*

Example 4. Now consider $S^3 \subset \mathbb{R}^4$ where we think of $\mathbb{R}^4 = \mathbb{H}$ as the quaternions. Recall that these are

$$\mathbb{H} = \{a + ib + jc + kd \mid a, b, c, d \in \mathbb{R}\}$$

with the rules that:

$$i^2 = j^2 = k^2 = -1 \qquad ij = k \qquad jk = i \qquad ki = j$$

Note that the unit is just 1 and the inverse of $a + bi + cj + dk$ is

$$\frac{a - bi - cj - dk}{a^2 + b^2 + c^2 + d^2}$$

Now if we look at $S^3 \hookrightarrow \mathbb{H}$, it turns out that we have the following:

Exercise 2. Check that S^3 is closed under quaternionic multiplication.

So S^3 is a Lie group because of the quaternions.

Example 5. As a Lie group, $\mathbb{C}^\times \simeq S^1 \times \mathbb{R}_{>0}$ where we send $re^{i\theta}$ to (θ, r) and similarly, $\mathbb{H}^\times \simeq S^3 \times \mathbb{R}_{>0}$. Note that $\mathbb{C}^\times$ and $\mathbb{H}^\times$ are noncompact. Note that this has nothing to do with the group theory, but rather the geometry. Similarly, we can talk about abelian Lie groups, but this is independent of the manifold structure.

Remark 2. None of the other spheres besides S^0, S^1, and S^3 are Lie groups. We might expect S^7 to be a Lie group on account of the so-called octonions, but the octonions do not form an associative algebra.

Example 6. S^2 is not a Lie group. Recall that $\chi\left(S^2\right) = 2$. But we also have the following:

Claim 1. If G is a connected Lie group, $\chi(G) = 0$. If G is finite, $\chi(G) = |G|$.

Example 7. Vector spaces and their automorphisms provide some nice[1] examples. Let V be a finite dimensional vector space, then the general linear group:

$$\mathrm{GL}(V) = \mathrm{GL}(n, \mathbb{R}) = \mathrm{Aut}(V)$$

is the set of $n \times n$ invertible matrices.

Now recall that $\mathrm{GL}(V)$ also acts on $\Lambda^{\dim V} V \simeq \mathbb{R}$, and we get the following

$$\begin{array}{ccc} \mathrm{GL}(V) & \circlearrowright & V \\ \big\downarrow{\scriptstyle \det} & & \\ \mathrm{GL}\left(\Lambda^{\dim V} V\right) & \circlearrowright & \Lambda^{\dim V} V \simeq \mathbb{R} \end{array}$$

so we get a short exact sequence:

$$1 \to \mathrm{SL}(V) \to \mathrm{GL}(V) \to \mathrm{GL}\left(\Lambda^{\dim V} V\right) \to 1$$

So the reason some people "don't like" $\mathrm{GL}(V)$ is that it has a normal subgroup, so it is not simple. Note that it splits as a manifold, but not as a group. But is $\mathrm{SL}(V)$ simple? If a group has a nontrivial center, then it certainly isn't simple. And the center of $\mathrm{SL}(V)$ has the diagonal matrices

$$\begin{pmatrix} a & \cdots & 0 \\ \cdots & a & \cdots \\ 0 & \cdots & a \end{pmatrix}$$

where $a^n = 1$ as its center.

Exercise 3. Show that $[\mathrm{SL}(V), \mathrm{SL}(V)] = \mathrm{SL}(V)$.

Fact 1. *All normal subgroups of* $\mathrm{SL}(V)$ *are finite, and in fact are contained in the center.*

[1] According to Professor Nadler, humans understand linear things and basically nothing else.

2.1. **Group actions.** Group actions are the main reason Lie was interested in such things to begin with.

Definition 3. Given a Lie group G and a manifold X, an action of G on X is a smooth map $a : G \times X \to X$ such that

(1) a is associative:
$$a\left(g_1, a\left(g_2, x\right)\right) = a\left(m\left(g_1, g_2\right), x\right)$$
(2) This is unital, so $a\left(e, x\right) = x$.

The two examples to think about are the following:

Example 8. Suppose X is a vector space, and every element of G acts by not only a smooth map, but a linear map. Then this action is called a *representation* and we think of this as a homomorphism $G \to \mathrm{GL}\left(X\right)$.

Exercise 4. Understand the natural action of $\mathrm{SL}\left(2, \mathbb{C}\right)$ on $\mathbb{CP}^1$.